Math 2130. Lecture 7.

02.01.2021

Last time. Spanning properties via echelon forms:

Let $\{\vec{v}_1, \ldots, \vec{v}_k\} \subseteq \mathbb{R}^n$ and let $C = \left[\vec{v}_1 \middle| \vec{v}_2 \middle| \cdots \middle| \vec{v}_k\right]$. Then

(1). We have $\vec{v} \in \mathrm{Span}\{\vec{v}_1, \ldots, \vec{v}_k\}$ for a given vec. $\vec{v} \in \mathbb{R}^n$ if and only if the matrix $A = [C \mid \vec{v}]$ doesn't have a pivot in the last column, i.e., EF(A) has no row of the form $[0 \cdots 0\ b]$ where $b \neq 0$.

(2) We have $\vec{v} \in \mathrm{Span}\{\vec{v}_1, \ldots, \vec{v}_k\}$ for all vectors $\vec{v} \in \mathbb{R}^n$ if and only if EF(A) has no zero rows, i.e., A has a pivot in every row.

Today. Linear independence.

Let $S = \{\vec{v}_1, \dots, \vec{v}_k\} \in \mathbb{R}^n$.

Non Eg: This doesn't have to be the case:

Take $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, then

$2 \cdot \vec{v}_1 - 1 \cdot \vec{v}_2 = \boxed{0}$, so $\{\vec{v}_1, \vec{v}_2\}$ is not lin. ind.

Def We say S is linearly independent or $\vec{v}_1, \dots, \vec{v}_k$ are linearly independent

if $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k = 0$ only when $c_1 = c_2 = c_3 = \dots = 0$.

If S is not lin. ind., we say S is linearly dependent.

Def. (trivial lin. comb.) The linear comb. $0 \cdot \vec{v}_1 + \dots + 0\vec{v}_k$ is called

the trivial lin comb. of $\vec{v}_1, \dots, \vec{v}_k$.

certainly this equals 0.

So, a set of vectors is lin ind. iff the only lin comb. of them that equals zero is the trivial one.

Q: Can we determine whether S is lin. ind. via some echelon forms? **A:** Yes.

As before, we start with reinterpreting linear ind. via matrix equations:

To say "$c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = 0 \implies c_1 = c_2 = \cdots = c_k = 0$" is the same as saying "the vector equation $x_1\vec{v}_1 + \cdots + x_k\vec{v}_k = 0$, (or equivalently the matrix $\left[\vec{v}_1 \middle| \vec{v}_2 \middle| \cdots \middle| \vec{v}_k\right] \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix}}_{\vec{x}} = \vec{0}$) has only the trivial soln $x_1 = x_2 = \cdots = x_k = 0$ (or $\vec{x} = \vec{0}$) as its unique soln",

i.e., it's equivalent to saying $\left[\vec{v}_1 \middle| \vec{v}_2 \middle| \cdots \middle| \vec{v}_k\right]\vec{x} = \vec{0}$ has a unique soln.

Thm. Let $S = \{\vec{v_1}, \cdots, \vec{v_k}\} \subseteq \mathbb{R}^n$ and $C = [\vec{v_1} | \vec{v_2} | \cdots | \vec{v_k}]$. Then

TFAE:

(1) S is lin. ind.

(2) The vec eq. $x_1\vec{v_1} + \cdots + x_k\vec{v_k} = 0$ has the trivial soln as its only soln.

(3) The mat. eq. $[\vec{v_1} | \cdots | \vec{v_k}]\vec{x} = \vec{0}$ only has the trivial soln.

(4). $EF(C)$ "has no free variables", i.e., every column is pivot in C.

$\Updownarrow$

" C has a pivot in every column "

$\downarrow$

compare with Page 1.

Example.

(1) $S = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$ $\quad C = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$. Every column is pivot in

E.F.

C, therefore S is lin independent. $\left(x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightsquigarrow \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 2 & 4 & 0 \end{array}\right] \right.$

$$\longrightarrow \begin{array}{cc}
b & b \\
\left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & -2 & 0 \end{array}\right]
\end{array}$$

$\longrightarrow$ all variables are bars because every column in C is pivt so there's a unique soln.)

(2) $S = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix} \right\}$

Soln 1: $C = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$. The second column in C is not pivt,

$b \quad f$

so S is not linearly ind.

(is lin. dep.)

Soln 2: Note that $3 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \vec{0}$,

so S is not lin ind. $\longleftarrow$ perfectly good. one doesn't have to use the thm.

Some other criteria for linear ind.

Let $S = \{\vec{v}_1, \ldots, \vec{v}_k\} \in \mathbb{R}^n$.

(a). If $\vec{0} \in S$, then S is not lin ind.

(e.g. $S = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\} \to \underbrace{\underset{\substack{c_1 \\ \neq \\ 0}}{\underline{1}} \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \underset{c_2}{\underline{0}} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \underset{c_3}{\underline{0}} \begin{bmatrix} 3 \\ 4 \end{bmatrix}}_{\text{nontrivial lin. comb.}} = 0 \Rightarrow$

S is lin dep.)

More generally, say $\vec{v}_1 = \vec{0}$, then $1 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2 + \cdots + 0 \cdot \vec{v}_k = 0$ is a nontrivial lin comb of S which equals 0, so S is lin dep.

(b) If $|S| = 1$, i.e., $S = \{\vec{v}_1\}$ for a vector $\vec{v}_1$, then S is lin ind. iff $\vec{v}_1 \neq 0$.

why: if $\vec{v}_1 = 0$, then S is lin dep. by (a); if $\vec{v}_1 \neq 0$, then $\vec{v}_1 = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ where $a_i \neq 0$ for some $1 \leq i \leq n$. But then $c\vec{v}_1 = 0 \Rightarrow ca_i = 0 \to c = 0$. $\square$

(c). If $|S| = 2$, say $S = \{\vec{v_1}, \vec{v_2}\}$, then if $\vec{0} \in S$ then S is lin dep; if $\vec{0} \notin S$, then S is lin dep iff one elt in S is a multiple of the other.

↓

Pf: Say $0 \notin S$. If $\vec{v_2} = c\vec{v_1}$, then $-c\vec{v_1} + \vec{v_2} = 0$ with $c \neq 0$, so S is lin dep. Conversely, if S is lin dep, say $c_1\vec{v_1} + c_2\vec{v_2} = 0$ where $c_1 \neq 0$ or $c_2 \neq 0$. (a) if $c_1 \neq 0$, then $\vec{v_1} = -\frac{c_2}{c_1}\vec{v_2}$

(b) if $c_2 \neq 0$, then $\vec{v_2} = -\frac{c_1}{c_2}\vec{v_1}$. □

S is lin dep iff one elt in S is a multiple of the other.

e.g., $S_1 = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$. $\rightsquigarrow$ $\vec{v_1} = 0 \cdot \vec{v_2}$ $\Rightarrow$ S is lin dep.

$\quad\quad\quad\quad \vec{v_1} \quad\; \vec{v_2}$

$S_2 = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\} \to$ $\vec{v_2}$ is not a mult of $\vec{v_1}$ $\Rightarrow$ S is lin ind

$\vec{v_1}$ .. - . -, $\vec{v_2}$

$S_3 = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix} \right\} \to \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\Rightarrow$ S is lin dep.

(d). **Thm:** A set $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\} \subseteq \mathbb{R}^n$ is linearly dependent iff no elt in S is a lin comb of the other elts of S.

E.g. (a) $S_1 = \left\{ \underset{\vec{v}_1}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}, \underset{\vec{v}_3}{\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}} \right\}$. $\vec{v}_3 = \vec{v}_1 + 2\vec{v}_2$ by inspection,

So S_1 is not lin ind.

(b) $$\begin{array}{c} \begin{bmatrix} 4 & 1 & 0 \\ -7 & 5 & 3 \\ -3 & 3 & 3 \end{bmatrix} \\ \vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3 \end{array} \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$\rightsquigarrow \vec{v}_1 + 2\vec{v}_2 - \vec{v}_3 = 0 \implies \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is lin. dep.

eq. $\vec{v}_3 = \vec{v}_1 + 2v_2$

Upshot: If $\vec{0} \in S$, $|S| \leq 2$, or you can easily notice a dep. relation, use the shortcuts or def to test lin ind/dep.

If not, use EF and the main thm (look for non-pivot cols).

Next time:
- geometry of vectors
- homogeneous vs. nonhomogeneous eqs.