Math 2130. Lecture 6.

Last time:

- p.v.f. of LES solns.
- LES as vector and matrix equations.
- Span membership via LES:

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ be a set of vectors in $\mathbb{R}^n$.

Q1. Let $\vec{v} \in \mathbb{R}^n$. Question: Is $\vec{v}$ in the span of $\vec{v}_1, \ldots, \vec{v}_k$?

Today:

- two related / similar questions

— Spanning sets of $\mathbb{R}^n$. Q2

— linear independence. Q3

Goal: state Q_1, Q_2, Q_3 and find answers via echelon form.

Q1. Span membership

E.g. Take $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} \in \mathbb{R}^3$ and $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in \mathbb{R}^3$.

Note: The following questions are all equivalent:

$\boxed{\text{Is } \vec{v} \text{ in } \mathrm{Span}(\{v_1, v_2\})\text{?}}$ $\Leftrightarrow$ Do there exist $x_1, x_2 \in \mathbb{R}$ s.t. $\vec{v} = x_1 \cdot \vec{v}_1 + x_2 \vec{v}_2 = x_1 \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$? $\Leftrightarrow$ Does the vec. eq. $x_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ or equiv. the matrix eq: $\begin{bmatrix} 1 & 2 \\ 2 & 5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, have a soln?

$\Leftrightarrow$ Is the LES

$$\begin{cases} x_1 + 2x_2 = 1 \\ 2x_1 + 5x_2 = 1 \\ x_2 = 1 \end{cases}$$

consistent?

We know the answer!

"watch out for a bad row $[0 \; 0 \cdots 0 \; b]$" $b \neq 0$.

More generally, ...

Thm 1. Let $A = \left[\, C \mid \vec{v} \,\right]$ where $C = \left[\, \vec{v}_1 \mid \vec{v}_2 \mid \cdots \mid \vec{v}_k \,\right]$.

Then the following are equivalent (T.F.A.E.):

(a) $\vec{v} \in \mathrm{Span}\left(\{\vec{v}_1, \dots, \vec{v}_k\}\right)$

(b) The vec equation $x_1\vec{v}_1 + \cdots + x_k\vec{v}_k = \vec{v}$ has a soln

(c) The matrix eq. $C \cdot \vec{x} = \vec{v}$ has a soln

(d) The LES with aug. matrix A has at least one soln.

(e). Some echelon form of A has no row of the form $[0\ 0 \cdots 0\ b]$ where $b \neq 0$.

$\Updownarrow$

(e') The last column of A is pivot.

Eg. Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} \in \mathbb{R}^3$ and $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in \mathbb{R}^3$.

<u>Q:</u> $\vec{v} \in \text{Span}\{\vec{v}_1, \vec{v}_2\}$?

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 1 \\ 0 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

Since the EF of $\left[\vec{v}_1 \,\middle|\, \vec{v}_2 \,\middle|\, \vec{v}\right]$ has the last column as a pivot column, $\vec{v} \notin \text{Span}\{\vec{v}_1, \vec{v}_2\}$.

Q2. Spanning properties

Def (Spanning sets) A set of vectors $S=\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ in $\mathbb{R}^n$ is called a spanning set (of $\mathbb{R}^n$) if every elt. $\vec{v}$ in $\mathbb{R}^n$ is in its span

Examples/Nonexamples

(0). In $\mathbb{R}^2$, $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ is a spanning set: $\forall a,b$, $\begin{bmatrix}a\\b\end{bmatrix}=a\begin{bmatrix}1\\0\end{bmatrix}+b\begin{bmatrix}0\\1\end{bmatrix}$.

(1). By the last page, $\vec{v}_1=\begin{bmatrix}1\\2\\0\end{bmatrix}$ and $\vec{v}_2=\begin{bmatrix}2\\5\\1\end{bmatrix}$ don't span $\mathbb{R}^3$: at least one elt, namely $\vec{v}=\begin{bmatrix}1\\1\\1\end{bmatrix}$, is not in their span.

(2). In $\mathbb{R}^2$, $S=\left\{\begin{bmatrix}2\\1\end{bmatrix}\right\}$ is not a spanning set; e.g., $\begin{bmatrix}2\\2\end{bmatrix}$ is not in its span: $\begin{bmatrix}2\\2\end{bmatrix}=x\begin{bmatrix}2\\1\end{bmatrix} \Rightarrow \begin{bmatrix}2\\2\end{bmatrix}=\begin{bmatrix}2x\\x\end{bmatrix} \Rightarrow \begin{cases}x=1\\x=2\end{cases}$. impossible.

Statement of Q2: Let $S = \{\vec{v}_1, \dots, \vec{v}_k\} \subseteq \mathbb{R}^n = \left\{ \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} : a_1, a_2, \dots, a_n \in \mathbb{R} \right\}$

Reinterpretation: S is a spanning set

$\Leftrightarrow$ $\vec{v} \in \operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$ $\forall \vec{v} \in \mathbb{R}^n$ $\Leftrightarrow$ The conditions in Thm 2 is true $\forall \vec{v} \in \mathbb{R}^n$.

Thm 2: Let $C = \left[\vec{v}_1 \middle| \cdots \middle| \vec{v}_k \right]$. Then TFAE:

(a) S is a spanning set.

(b) The vec. eq. $x_1 \vec{v}_1 + \cdots + x_k \vec{v}_k = \vec{v}$ has a soln $\forall \vec{v} \in \mathbb{R}^n$.

(c) The mat. eq. $C \cdot \vec{x} = \vec{v}$ has a soln $\forall \vec{v} \in \mathbb{R}^n$

(d). The LES with aug. matrix $\left[C \middle| \vec{v} \right]$ is consistent $\forall \vec{v} \in \mathbb{R}^n$.

(e) $EF(A)$ has no zero row $\leftrightarrow$ (e') Every row in A is pivot.

Note: Compare Thm 1 with Thm 2. Note that we've argued (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) but not (d) $\Leftrightarrow$ (e).

Example. (1) Do $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$ span $\mathbb{R}^3$?

i.e., Is $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ a spanning set of $\mathbb{R}^3$?

Soln:

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}$$

Since EF(A) has a zero row, S does not span $\mathbb{R}^3$.

(2). $$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$$ $\to$ echelon, no zero row.

So $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$ is a spanning set in $\mathbb{R}^2$.

The "reason" why (d) $\Leftrightarrow$ (e).

Given $\vec{v} \in \mathbb{R}^n$

$$EF\Big(\overbrace{\left[\vec{v}_1 \middle| \vec{v}_2 \middle| \cdots \middle| \vec{v}_k \middle| \vec{v}\right]}^{A_{\vec{v}}}\Big) = EF\left(\left[C \middle| \vec{v}\right]\right)$$

$$= \left[\begin{array}{c|c} EF(C) & \begin{matrix} * \\ * \\ \vdots \\ * \end{matrix} \end{array}\right].$$

If $EF(C)$ has a zero row, then $EF(A_{\vec{v}}) = \left[\begin{array}{c|c} EF(C) & \begin{matrix} * \\ \vdots \\ * \end{matrix} \\ 0 \ 0 \cdots 0 & \end{array}\right]$

has a bad row $[0 \ 0 \cdots 0 \mid b]$ for some choice of $\vec{v}$, which implies $\vec{v} \notin \text{Span}(S)$.

Next time: linear independence of a set of vecs. and echelon criterion for lin ind.