Math 2130. Lecture 5.

<u>Last time</u>:

- Number of solns for a LES from the echelon form of its augmented matrix.

e.g. $\begin{bmatrix} 0 & 3 & 7 \\ 2 & 6 & 10 \end{bmatrix} \longrightarrow \begin{array}{c} \begin{bmatrix} 2 & 6 & 10 \\ 0 & 3 & 7 \end{bmatrix} \\ \begin{array}{cc} b & b \end{array} \end{array}$ → consistent, unique soln.

- To get the exact sols, use REF.

$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & \frac{7}{3} \end{bmatrix} \longrightarrow \begin{cases} x = -2 \\ y = \frac{7}{3} \end{cases}$

- Linear comb. of vectors. e.g. $3 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 2 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $x \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} - y \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k$.

Today. New notation and terminology.

1. Parametric vector form for solns of LES.

If an LES has inf. solns. we can write the solns as a linear combination of constant vectors where all but possibly one coeff is a free variable.

This is called the parametric vector form. (p.v.f.)

e.g.
$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \longrightarrow \begin{bmatrix} z-2 \\ 3-2z \\ z \end{bmatrix} = \begin{bmatrix} z \\ -2z \\ z \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix} = \underbrace{z \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}}_{\text{p.v.f.}}$$

free variable $\longleftrightarrow$ parameter

Ex. Find the soln set of $\begin{cases} x - 3y - 5z = 0 \\ y - z = -1 \end{cases}$ in p.v.f.

Soln: The augmented matrix is $A = \left[\begin{array}{ccc|c} 1 & -3 & -5 & 0 \\ 0 & 1 & -1 & -1 \end{array}\right]$ $+$ $\times 3$

Gaussian elimination produces

$$A \to \begin{array}{c} \left[\begin{array}{ccc|c} 1 & 0 & -8 & -3 \\ 0 & 1 & -1 & -1 \end{array}\right] \\ \begin{array}{ccc} b & b & f \end{array} \end{array}$$

↖ reduced echelon, so the soln set is

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : \begin{array}{l} x - 8z = -3 \\ y - z = -1, \end{array} \; z \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : \begin{array}{l} x = -3 + 8z \\ y = -1 + z \\ z \in \mathbb{R} \end{array} \right\} = \left\{ \begin{bmatrix} -3+8z \\ -1+z \\ z \end{bmatrix} : z \in \mathbb{R} \right\}$$

$$= \left\{ \boxed{\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 8 \\ 1 \\ 1 \end{bmatrix} : z \in \mathbb{R}} \right\},$$ with the boxed part being the p.v.f. □

2. Vector equations.

Since a vector stores multiple numbers at a time, we can express each LES (multiple equations) as a vector equation, i.e., an equation of the form $x_1\vec{v_1} + x_2\vec{v_2} + \cdots + x_n\vec{v_n} = \vec{v}$.

e.g.
$$\begin{cases} 2x = 10 \\ x + 5y = 20 \end{cases} \longleftrightarrow \begin{bmatrix} 2x \\ x+5y \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$$

$$\longleftrightarrow x \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{\vec{v}_1} + y \underbrace{\begin{bmatrix} 0 \\ 5 \end{bmatrix}}_{\vec{v}_2} = \underbrace{\begin{bmatrix} 10 \\ 20 \end{bmatrix}}_{\vec{v}}.$$

variables in the LES

$\updownarrow$

the coefs in the vec. eq.

Another example.

the vector in the vec. eq $\longleftrightarrow$ columns in the aug. matrix

$$x_1\begin{bmatrix}3\\1\\4\end{bmatrix}+x_2\begin{bmatrix}-1\\2\\3\end{bmatrix}=\begin{bmatrix}10\\1\\0\end{bmatrix} \quad\Leftrightarrow\quad \begin{cases}3x_1-x_2=10\\ x_1+2x_2=1\\ 4x_1+3x_2=0\end{cases} \quad\Leftrightarrow\quad \begin{bmatrix}3&-1&10\\1&2&1\\4&3&0\end{bmatrix}$$

Vector equation — L.E.S. — encoding matrix.

Summary: an L.E.S. with encoding matrix $A=\left[\begin{array}{c|c} C & \vec{b}\end{array}\right]$ can be rewritten as the vector equation $x_1\vec{v}_1+x_2\vec{v}_2+\cdots+x_n\vec{v}_n=\vec{b}$ where $\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_n$ are the columns of C, i.e., if $C=\left[\begin{array}{c|c|c|c}\vec{v}_1&\vec{v}_2&\cdots&\vec{v}_n\end{array}\right]$.

3. Matrix - vector products.

Let $A = \begin{array}{c} \left[\begin{array}{c|c|c|c} \begin{matrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{matrix} & a_{12} & \cdots & \begin{matrix} a_{1n} \\ \vdots \\ a_{mn} \end{matrix} \end{array}\right] \\ \begin{matrix} \vec{v}_1 & \vec{v}_2 & \quad & \vec{v}_n \end{matrix} \end{array}$ be a matrix and let $\vec{v} = \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix}$ be a vector.

We can sometimes (not always) define a product $A \cdot v$:

Def: If $n = p$, i.e., if # cols of A = # entries in $\vec{v}$,

then we define $A \cdot v$ to be the vector (mat × vec → vec)

$$A \cdot v = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = x_1 \vec{v}_1 + \cdots + x_n \vec{v}_n.$$

This has n entries ; see next page.

e.g.,

$$\begin{bmatrix} 2 & 4 & 0 \\ 5 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$ is not defined.

$$\underbrace{\begin{bmatrix} 2 & 4 & 0 \\ 5 & 1 & 2 \end{bmatrix}}_{2\times 3}_{A} \cdot \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{v} = 1\cdot\begin{bmatrix} 2 \\ 5 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2\cdot 1 + 4\cdot 2 + 0\cdot 3 \\ 5\cdot 1 + 1\cdot 2 + 2\cdot 3 \end{bmatrix} = \begin{bmatrix} 10 \\ 13 \end{bmatrix}$$

size 2

$$3\cdot\begin{bmatrix} 2 \\ 1 \end{bmatrix} - 4\cdot\begin{bmatrix} 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

$$x\begin{bmatrix} 2 \\ 1 \end{bmatrix} + y\begin{bmatrix} 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

Another way of understanding matrix-vector product:

$$\begin{bmatrix} \cdots \vec{r}_1 \cdots \\ \cdots \vec{r}_2 \cdots \\ \vdots \\ \cdots \vec{r}_m \cdots \end{bmatrix} \cdot \vec{v} = \begin{bmatrix} \vec{r}_1\cdot\vec{v} \\ \vec{r}_2\cdot\vec{v} \\ \vdots \\ \vec{r}_m\cdot\vec{v} \end{bmatrix}$$

$\cdot$ = inner product

4. Matrix equations.

Since we can write linear combinations as mat-vec. products and since each LES can be written as a vector equation $x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{v}$, we can write LES as matrix equations, that is, equations of the form $C \cdot \vec{x} = \vec{v}$.

e.g.

$$\begin{cases} x + 2y = 5 \\ -x + 10y = 3 \end{cases} \Longleftrightarrow x\begin{bmatrix} 1 \\ -1 \end{bmatrix} + y\begin{bmatrix} 2 \\ 10 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix} \Longleftrightarrow \underbrace{\begin{bmatrix} 1 & 2 \\ -1 & 10 \end{bmatrix}}_{C} \cdot \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 5 \\ 3 \end{bmatrix}}_{\vec{v}}$$

$$\text{Aug. matrix } A = \begin{bmatrix} 1 & 2 & 5 \\ -1 & 10 & 3 \end{bmatrix}$$

<u>Consequence.</u> We can write each LES in three ways:

① Traditional

<u>e.g.</u>

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ \vdots \qquad\qquad\qquad \vdots \\ a_{m1}x_1 + \cdots\cdots + a_{mn}x_n = b_m \end{cases}$$

$$\Updownarrow$$

$$A = \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right] = \left[C \,\middle|\, \vec{b} \right]$$

② as a vec. eq.

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

③ matrix eq.

$$C\vec{x} = \vec{b}$$

More generally, an L.E.S. with augmented matrix $A = [C \mid \vec{b}]$ can be written

a) the matrix eq. $C \cdot \vec{x} = \vec{b}$ where $\vec{x}$ stands for $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

5. Spans of vectors.

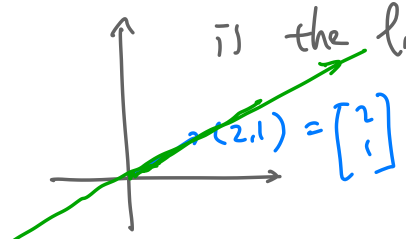
Def. Let $S=\{\vec{v}_1,\dots,\vec{v}_k\}$ be a set of vectors in $\mathbb{R}^n$.
The **span** of S (or of $\vec{v}_1,\dots,\vec{v}_k$) is the set

$$\mathrm{Span}(S) := \{c_1\vec{v}_1 + \dots + c_k\vec{v}_k \mid c_1, c_2, \dots, c_k \in \mathbb{R}\}$$

of all linear comb. of $\vec{v}_1,\dots,\vec{v}_k$.

E.g. · In $\mathbb{R}^2$ ( ), $\mathrm{Span}\left(\left\{\begin{bmatrix}2\\1\end{bmatrix}\right\}\right) = \left\{c\begin{bmatrix}2\\1\end{bmatrix} : c\in\mathbb{R}\right\} = \left\{\begin{bmatrix}2c\\c\end{bmatrix} : c\in\mathbb{R}\right\}$

is the line going through the origin and the point $\begin{bmatrix}2\\1\end{bmatrix}$.



→ More on geometry of vectors later.

Span and the consistency problem.

Consider $\vec{v} = \begin{bmatrix} 4 \\ 8 \\ 12 \end{bmatrix}$ and $S = \left\{ \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Q: Is $\vec{v}$ in Span(S) ?

Claim: The question is equivalent to asking if a certain LES is consistent.

HW: What's the LES and its vec. eq / mat. eq. forms ?

What's the answer to the question ?