Math 4140. Lecture 41. Review for Ch. 5 & 6 available online. 04. 26. 2021.

Next lecture: review.

Last time: · orthogonal / orthonormal bases of $\mathbb{R}^n$

advantage: makes coordinate vectors easier to compute.

If $B = \{v_1, \dots, v_n\}$ is an orthogonal basis, then $\forall v \in \mathbb{R}^n$

we have $[v]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ is given by $c_i = \dfrac{\langle v, v_i \rangle}{\langle v_i, v_i \rangle}$.

$\langle v_i, v_i \rangle$ $=$ $\langle v, v_i \rangle$ if B is orthonormal

· orthogonal projections

$y \in \mathbb{R}^n$, orthogonal basis $B = \{v_1, \dots, v_n\}$ of $\mathbb{R}^n$, a subset I of B

$\hat{y}_I$, proj of y onto the span of B. $\hat{y}_I = \sum_{v_i \in I} c_i v_i$.

geometric properties of $\hat{y}_I$: $\hat{y}_I \in \operatorname{span} I$, $\hat{y}_I$ is the closest point in $\operatorname{span} I$ to y.

Today:

- more on orthogonal projections
- the Gram-Schmidt process (for constructing orthogonal bases)

1. Orthogonal projections, revisited.

Facts:

- As we'll see, every subspace W of $\mathbb{R}^n$ has at least one orthogonal basis.
- Any basis of $W \leq \mathbb{R}^n$ (subspace) can be extended to a basis of $\mathbb{R}^n$.

e.g. $W =$ xy-plane $= \mathrm{Span}\{e_1, e_2\} \leq \mathbb{R}^3$ extend $\{e_1, e_2, e_3\}$, basis of $\mathbb{R}^3$.

E.g. Take $B = \left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}}_{v_2}, \underbrace{\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}}_{v_3} \right\} \subseteq \mathbb{R}^3$ and $W = \mathrm{Span}\{v_1, v_2\}$.

Then $\left\{ u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is an ortho. basis of W, and it can be extended to the standard basis of $\mathbb{R}^3$.

Point: We may talk about the orthogonal projection ($\mathrm{proj}_W y$) of $y \in \mathbb{R}^n$ onto a subspace $W \subseteq \mathbb{R}^n$ without having a given orthogonal basis of $\mathbb{R}^n$ or of W.

In the previous example, if $y = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$, then we can speak of $\mathrm{proj}_W y$ even if we are not given an orthogonal basis of W to start with.

The <u>distance from y to W</u> is still $\| y - \mathrm{proj}_W y \|$.

$y = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$, $W = \mathrm{Span}\left\{ \underbrace{\underset{v_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}}}_{\text{not orthogonal}} \right\}$ $\rightsquigarrow$ * we can compute an orthogonal basis $\left\{ u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ of W

$\rightarrow$ $\mathrm{proj}_W y = c_1 u_1 + c_2 u_2$, $c_i = \dfrac{\langle y, u_i \rangle}{\langle u_i, u_i \rangle}$ for $i = 1, 2$.

Remark: As noted above, to find $\mathrm{proj}_W y$ we often need to find an orthogonal basis of W ourselves. $\longrightarrow$ done by the Gram-Schmidt process.

2. The Gram-Schmidt algorithm.

Input: A basis $B = \{x_1, x_2, \dots, x_p\}$ of a subspace W of $\mathbb{R}^n$.

Output: An orthogonal basis $B' = \{v_1, v_2, \dots, v_p\}$ of W.

Main idea: repeatedly compute vectors of the form $y - \text{proj}_u\, y$ where $y \in B$ and u is a subspace of W.

The algorithm:

(1) initial step: Set $v_1 = p_1$

e.g.

$$\left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{x_1}, \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}}_{x_2}, \underbrace{\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}}_{x_3} \right\} \longrightarrow v_1 = x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

(2). when $\{v_1, v_2, \dots, v_{i-1}\}$ have been found, we set

$U_{i-1} = \text{Span}\{v_1, \dots, v_{i-1}\}$ and compute

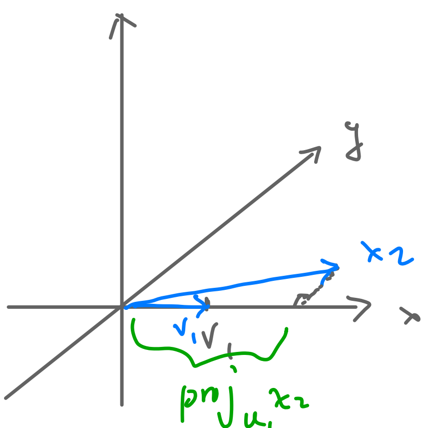
$$v_i = x_i - \text{proj}_{U_{i-1}} x_i .$$

e.g. $\{v_1\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$, $U_{1-1} = \text{Span}\{v_1\}$.

$$\rightarrow \quad v_2 = x_2 - \text{proj}_{U_{i-1}} x_2 = x_2 - \frac{\langle x_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$$

$$= \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \frac{\left\langle \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\rangle}{\left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\rangle} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$



Now $\left\{ v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ are known, so we can compute v_3.

$$v_3 = x_3 - \text{proj}_{u_2} x_3 = x_3 - \text{proj}_{\text{span}\{v_1, v_2\}} x_3$$

$$= x_3 - \frac{\langle x_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle x_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2$$

$$= \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} - \text{proj}_{\text{span}\{e_1, e_2\}} \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}.$$

13) Stop when all $v_1, \cdots, v_p$ have been computed.

e.g. $\{v_1, v_2, v_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \right\}$ is a basis for W. $\square$

Example.

$$B = \left\{ \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 8 \\ 5 \\ 6 \end{bmatrix} \right\}. \qquad W = \operatorname{Span} B \subseteq \mathbb{R}^3.$$

$$\begin{array}{cc} x_1 & x_2 \end{array}$$

Soln: v_1 : $\quad v_1 = x_1 = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$.

v_2 : use $\{v_1\}$ and x_2. $\quad v_2 = x_2 - \operatorname{proj}_{\operatorname{Span}\{v_1\}} x_2$

$$= x_2 - \frac{\langle x_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$$

$$= \begin{bmatrix} 8 \\ 5 \\ 6 \end{bmatrix} - \frac{18}{10} \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 \\ 5 \\ 6 \end{bmatrix} - \begin{bmatrix} 27/5 \\ 0 \\ -9/5 \end{bmatrix} = \begin{bmatrix} 13/5 \\ 5 \\ 39/5 \end{bmatrix}.$$

$\Uparrow$

So $\left\{ v_1 = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} 13/5 \\ 5 \\ 39/5 \end{bmatrix} \right\}$ is an orthogonal basis of W.

Check: $\langle v_1, v_2 \rangle = 0$. ✓

$$B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} \right\}. \quad W = \operatorname{Span} B \subseteq \mathbb{R}^3. \quad (\text{so } W = \mathbb{R}^3)$$

$$x_1 \qquad x_2 \qquad x_3$$

Soln:

(1) $v_1 = x_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$

(2) Now that $\{v_1\}$ is known, we can compute v_2.

$$v_2 = x_2 - \operatorname{proj}_{\operatorname{Span}\{v_1\}} x_2 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} - \frac{\langle x_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$$

$$= \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} - \frac{5}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 4 \end{bmatrix}.$$

Check: $v_1 \perp v_2$

Now that $\{v_1, v_2\}$ are known, we can compute v_3:

$$v_3 = x_3 - \underset{\substack{u_2 \\ \swarrow \\ \text{span}\{v_1, v_2\}}}{\text{proj}}\, x_3 = x_3 - \frac{\langle x_3, v_1\rangle}{\langle v_1, v_1\rangle} v_1 - \frac{\langle x_3, v_2\rangle}{\langle v_2, v_2\rangle} v_2$$

$$= \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} - \frac{6}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{(-2)}{\frac{33}{2}}\begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} + \frac{4}{33}\begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{2}{33} \\ \frac{2}{33} \\ \frac{16}{33} \end{bmatrix} = \begin{bmatrix} 64/33 \\ -64/33 \\ 16/33 \end{bmatrix}.$$

So $\{v_1, v_2, v_3\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 4 \end{bmatrix}, \begin{bmatrix} 64/33 \\ -64/33 \\ 16/33 \end{bmatrix} \right\}$ is an orthogonal basis of W.