Math 2130. Lecture 40.

04. 23. 2021.

Last time:

- normalization of vectors $0 \neq v \longmapsto \dfrac{v}{\|v\|} \to$ unit vector
- orthogonal vectors : $u \perp v \iff u \cdot v = 0$.
- Prop: An orthogonal set of nonzero vectors $\{v_1, \dots, v_k\}$ is automatically lin ind.

Pf idea: e.g. say $k=3$. $av_1 + bv_2 + cv_3 = 0$

to show, say, $b=0$, take the inner prod with v_2.

$$\underbrace{a v_1 \cdot v_2}_{0} + \underbrace{\boxed{b\, v_2 \cdot v_2}}_{v_2 \cdot v_2 \neq 0} + c\, \underbrace{v_3 \cdot v_2}_{0} = 0 \implies b = 0.$$

Today:

- orthogonal and orthonormal bases
- orthogonal projections

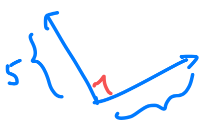
1. Orthogonal basis

Def. An orthogonal basis of $\mathbb{R}^n$ is a basis of $\mathbb{R}^n$ which is an orthogonal set.

An orthonormal basis of $\mathbb{R}^n$ is an orthogonal basis consisting of unit vectors.

Eg. $n=2$.

- the standard basis $B_1 = \{e_1, e_2\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is an orthonormal basis of $\mathbb{R}^2$.



len : 5

- the set $B_2 = \left\{ \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \end{bmatrix} \right\}$ is $\underline{\text{orthogonal}}$ (last lecture).

$\Downarrow$ B_2 is lin ind.

It has 2 elts, so B_2 is a basis and an orthogonal basis. However, it is not orthonormal.

$\downarrow$ normalize

- normalizing the elts in B_2 yields an orthonormal basis $B_2' = \left\{ \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}, \begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix} \right\}$.

Why are orthogonal bases interesting?

Recall: Given a basis $B = \{v_1, \dots, v_n\}$ of $\mathbb{R}^n$ and any $v \in \mathbb{R}^n$, we can ask for the coordinate vector $[v]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$. Finding $[v]_B$ is usually done by solving the equation $c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = v$.

For an orthogonal basis $C = \{v_1, \dots, v_n\}$, we can find the c_i's and hence $[v]_C$ more directly:

Thm. Let $C = \{v_1, \dots, v_n\}$ be an orthogonal basis of $\mathbb{R}^n$. Then for all $v \in \mathbb{R}^n$, we have

$$v = c_1 v_1 + \cdots + c_n v_n$$

where

$$c_i = \frac{\langle v, v_i \rangle}{\langle v_i, v_i \rangle}$$

} two inner. prod. comp. + one division.

$\Downarrow$

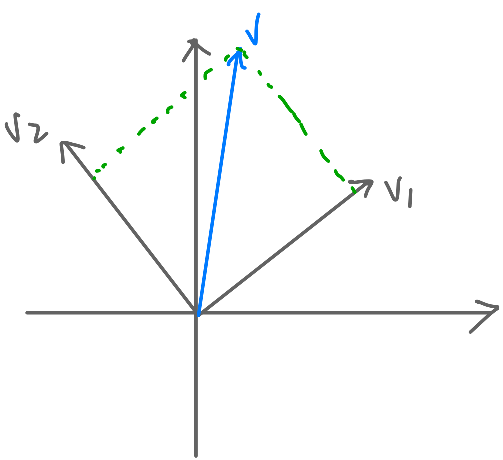
In particular, if C is orthonormal, then $c_i = \langle v, v_i \rangle$. $\longrightarrow$ straightforward.

Pf: Suppose $v = c_1 v_1 + \cdots + c_n v_n$. Let $1 \le i \le n$. "Dotting with" v_i on both sides gives

$$\langle v, v_i\rangle = \underbrace{c_1\langle v_1, v_i\rangle}_{=0} + \underbrace{c_2\langle v_2, v_i\rangle}_{=0} + \cdots + c_i\underbrace{\langle v_i, v_i\rangle}_{\text{only nonzero i.p.}} + \cdots + \underbrace{c_n\langle v_n, v_i\rangle}_{=0}$$

So $\langle v, v_i\rangle = c_i\langle v_i, v_i\rangle$, therefore $c_i = \dfrac{\langle v, v_i\rangle}{\langle v_i, v_i\rangle}$. □

Eg. $n=2$, $C = \left\{ \underset{v_1}{\begin{bmatrix}4\\3\end{bmatrix}}, \underset{v_2}{\begin{bmatrix}-3\\4\end{bmatrix}} \right\}$, $v = \begin{bmatrix}1\\6\end{bmatrix}$. To find $[v]_C$, we can



. (old way). solve $\underset{c_1}{x}\begin{bmatrix}4\\3\end{bmatrix} + \underset{c_2}{y}\begin{bmatrix}-3\\4\end{bmatrix} = \begin{bmatrix}1\\6\end{bmatrix}$.

E.X. check that $c_1v_1 + c_2v_2 = v$.

. (new way, since C is orthogonal):

$$c_1 = \frac{\left\langle \begin{bmatrix}1\\6\end{bmatrix}, \begin{bmatrix}4\\3\end{bmatrix} \right\rangle}{\left\langle \begin{bmatrix}4\\3\end{bmatrix}, \begin{bmatrix}4\\3\end{bmatrix} \right\rangle} = \frac{22}{25}, \quad c_2 = \frac{\left\langle \begin{bmatrix}1\\6\end{bmatrix}, \begin{bmatrix}-3\\4\end{bmatrix} \right\rangle}{\left\langle \begin{bmatrix}-3\\4\end{bmatrix}, \begin{bmatrix}-3\\4\end{bmatrix} \right\rangle} = \frac{21}{25}.$$

$$[v]_C = \begin{bmatrix}22/25\\21/25\end{bmatrix}.$$

2. Orthogonal projections

Let $\{v_1, \dots, v_n\}$ be an orthogonal basis of $\mathbb{R}^n$.

The decomp. $v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$

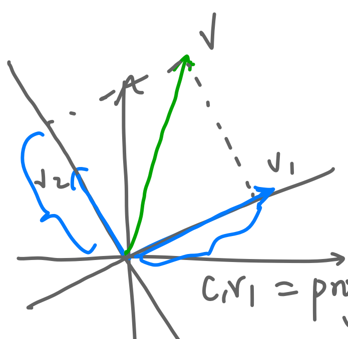
has geometric significance. For any $1 \le i \le n$, we call the

part/summand $c_i v_i = \frac{\langle v, v_i \rangle}{\langle v_i, v_i \rangle} \cdot v_i$ the projection of v onto

(the line spanned by) v_i; we denote $c_i v_i$ by $\mathrm{proj}_{v_i} v$.

E.g.:

$c_2 v_2 = \mathrm{proj}_{v_2} v$



$c_1 v_1 = \mathrm{proj}_{v_1} v$

Note: $\cdot\ \forall i$, $\mathrm{proj}_{v_i} v \in \mathrm{Span}\{v_i\}$.

$\cdot\ \forall i$, $v - \mathrm{proj}_{v_i} v \perp v_i$.

More generally ...

Def. (projection onto subspace) Let $C = \{v_1, v_2, \ldots, v_n\}$ be an orthogonal basis of $\mathbb{R}^n$. Let $y \in \mathbb{R}^n$. Consider the decomp

$$y = c_1 v_1 + \cdots + c_n v_n \quad \text{where} \quad c_n = \frac{\langle y, v_i \rangle}{\langle v_i, v_i \rangle}.$$

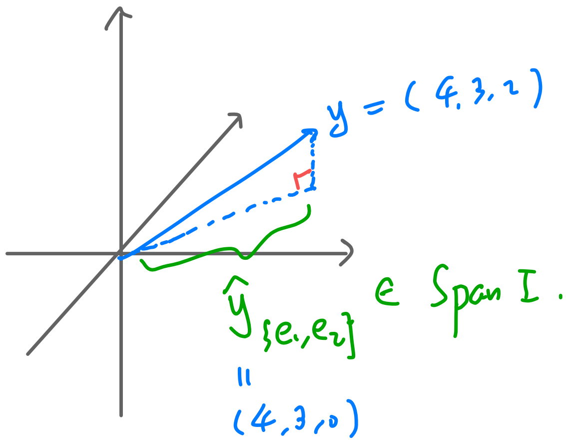
For any subset I of C, the linear combination

$$\hat{y}_I := \sum_{i : v_i \in I} c_i v_i$$

is called the orthogonal projection of y onto the span of I.

(On the previous page, we just took I to be a singleton set $\{v_i\}$.)

Eg. $n=3$. $C=\{e_1, e_2, e_3\}$ $\quad I=\{e_1, e_2\}$. $\rightarrow$ $\underbrace{\text{Span } I}_{\hat{y}_I}$ = the xy-plane.



$$y=\begin{bmatrix}4\\3\\2\end{bmatrix} = \underbrace{\overbrace{4e_1+3e_2}^{\hat{y}_I}}_{\hat{y}_{\{e_1,e_2\}}} + \overbrace{2e_3}^{y-\hat{y}_I}$$

Thm: Let $C, I, y, \hat{y}_I$ be as in the previous definition, then

(a). $y-\hat{y}_I \perp \hat{y}_I$

(b). $\forall w \in \text{Span } I$, we have $\|y-w\| \geq \|y-\hat{y}_I\|$.

(Thus, $\hat{y}_I$ is the closest point to y in Span I).

E.g. Let $C = \left\{ \overset{u_1}{\begin{bmatrix} 1 \\ 6 \end{bmatrix}}, \overset{u_2}{\begin{bmatrix} -12 \\ 2 \end{bmatrix}} \right\} \subseteq \mathbb{R}^2$.

(1) Verify that C is an orthogonal basis of $\mathbb{R}^2$:

$\longrightarrow$ Check $\langle u_1, u_2 \rangle = 0$. note that $|C| = 2$.

(2) Find $[v]_C$ for $v = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$. $\longrightarrow$ use $c_i = \dfrac{\langle v, v_i \rangle}{\langle v_i, v_i \rangle}$.

(3). Find the closest point to v on the line $\mathrm{Span}(u_1)$.

$\rightarrow$ get c_1, c_2 for $v = c_1 u_1 + c_2 u_2$.

the desired point is $c_1 u_1$.

(4). Find the shortest distance between v and a point $u \in \mathrm{Span}(u_1)$.

$\rightarrow$ compute $\| v - c_1 u_1 \|$.