Math 2130. Lecture 4.

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So far:
- elt. row ops. echelon forms
- using row ops to get Echelon forms.

The following picture is almost completed

$$\begin{tikzcd}[column sep=huge, row sep=large]
\text{given L.E.S} \arrow[r, "\text{encode}"] \arrow[d, dotted] & \text{matrix } A \arrow[d, "\text{Gaussian elimination}"] \\
\text{the soln} & \text{E.F or R.E.F. of } A. \arrow[l, "\text{decode/read off}"']
\end{tikzcd}$$

matrix A, the so-called **augmented matrix**. $= [\, C \mid \vec{b} \,]$

$$\begin{array}{cc} \text{coeff. matrix} & \text{scalar. column} \\ C & \vec{b} \end{array}$$

Today:
- Soln sets of L.E.S. from (reduced) echelon forms
- parametric vector forms.

1. Number of solns from echelon form (Is the system consistent? If so, how many solns?)

Definitions and observations.

1. Def. (Pivot columns, pivot positions, basic variables, free variables)

Let A be a matrix and B be any echelon form of A.

(e.g. $A = \begin{array}{c} \begin{array}{cccc} b & b & f & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{array}\right] \end{array} \longrightarrow B = \begin{array}{c} \begin{array}{cccc} p & p & & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$)

A pivot position in A is a position where there's a row-leading entry in B.

Such positions are independent of the choice of B.

A pivot row or column in A is a row or column in A with a pivot position.

For a L.E.S with augmented matrix $A = [\,C \mid \vec{b}\,]$, a variable is called a basic variable if its column is pivot and a free variable if its column is not pivot

Examples. (all from the last lecture)

(1) $\begin{cases} x+2y+3z=4 \\ 5x+6y+7z=8 \\ 9x+10y+11z=9 \end{cases} \longrightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \longrightarrow \begin{array}{cccc} x & y & z & \\ \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ b & b & f & \end{array}$

$$\text{Soln set} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : \begin{array}{l} x-z=-2 \\ y+2z=3 \end{array}, z\in\mathbb{R} \right\} = \left\{ \begin{bmatrix} z-2 \\ 3-2z \\ z \end{bmatrix} : z\in\mathbb{R} \right\}$$

Note: . There's at least one soln since there's no "nonsense rows" ($[0 \; 0 \; 0 \; b]$ $b\neq 0$).

i.e., the L.E.S is consistent.

. The L.E.S actually has inf. many solns since the variable z is free.

Rough idea: consistency question $\longleftrightarrow$ existence of nonsense rows; uniqueness when consistent $\longleftrightarrow$ existence of free variables

More examples.

Suppose these are aug. mat. of L.E.S.'s.

Recall the last matrices are the R.E.F.s.

: pivot

: not pivot

(a) $\begin{bmatrix} 1 & 3 & 7 \\ 2 & 2 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 7 \\ 0 & -4 & -8 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 7 \\ 0 & 1 & 2 \end{bmatrix} \longrightarrow \begin{array}{c} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \\ \begin{matrix} b & b & \end{matrix} \end{array}$

$\begin{cases} x = 1 \\ y = 2 \end{cases}$ no nonsense ("bad") row, no free var.

(b) $\begin{bmatrix} 1 & 3 & 7 \\ 2 & 6 & 14 \end{bmatrix} \longrightarrow \begin{array}{c} \begin{bmatrix} 1 & 3 & 7 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} b & f & \end{matrix} \end{array} \rightarrow \{x + 3y = 7\}$

no bad row $\exists$ free var.

(c). $\begin{bmatrix} 0 & 3 & 7 \\ 2 & 6 & 10 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 6 & 10 \\ 0 & 3 & 7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & \frac{7}{3} \end{bmatrix} \xrightarrow[R1 - 3R2]{R1 \uparrow} \begin{array}{c} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & \frac{7}{3} \end{bmatrix} \\ \begin{matrix} b & b & \end{matrix} \end{array} \rightarrow \begin{cases} x = -2 \\ y = \frac{7}{3} \end{cases},$

no bad row no free var.

(d) $\begin{bmatrix} 1 & 3 & 7 \\ 2 & 6 & 10 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 7 \\ 0 & 0 & -4 \end{bmatrix} \longrightarrow \begin{array}{c} \begin{bmatrix} 1 & 3 & 7 \\ 0 & 0 & 1 \end{bmatrix} \\ \begin{matrix} b & f & \end{matrix} \end{array} \rightarrow$

inconsistent, no soln. $\exists$ bad row

Prop 1. (Consistency via E.F.)

Let A be the aug. matrix of a L.E.S. , and let B be any echelon form of A. Then the L.E.S is consistent if and only if B has no row of the form $\left[\begin{array}{cccc|c} 0 & \cdots & 0 & 0 & b \end{array}\right]$ where b is a nonzero number.

Note: To tell if an L.E.S. is consistent, it suffices to get just any echelon form of its augmented matrix and not necessarily the R.E.F.

e.g. $\begin{bmatrix} 0 & 3 & 7 \\ 2 & 6 & 10 \end{bmatrix} \longrightarrow \underbrace{\begin{bmatrix} 2 & 6 & 10 \\ 0 & 3 & 7 \end{bmatrix}}_{E.F.}$ by elt. row ops, the corr. L.E.S is consistent (and has a unique soln).

3. <u>Prop 2.</u> (Number of solns from E.F.)

Say a L.E.S. has aug. matrix A. and assume that the L.E.S is consistent. Then the L.E.S has a unique soln iff all variables are basic, i.e., iff all variable columns in A are pivot.

Note: As before, here we don't need a R.E.F to tell the number of solns.

$\boxed{}$ stands for a nonzero number

E.g.

$$A \to \begin{bmatrix} \boxed{} & * & * & * \\ 0 & \boxed{} & * & * \\ 0 & 0 & 0 & 3 \end{bmatrix} \quad ; \quad A \to \begin{array}{c} \begin{bmatrix} \boxed{} & * & * & * & * \\ 0 & 0 & \boxed{} & * & * \\ 0 & 0 & 0 & \boxed{} & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccc} & f & & & \end{array} \end{array}$$

↓ inconsistent

→ consistent.
inf. many solns.

E.g: Transform the matrix $A = \begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & -1 & 3 & 1 \\ 4 & -3 & 0 & 3 \end{bmatrix}$ into Echelon form, determine if the L.E.S. encoded by A is consistent, then find the soln set of the L.E.S.

$\exists$ unique soln
$\Uparrow$

Soln: $A \longrightarrow \begin{bmatrix} 4 & -1 & 3 & 1 \\ 0 & 1 & 2 & -2 \\ 4 & -3 & 0 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 4 & -1 & 3 & 1 \\ 0 & 1 & 2 & -2 \\ 0 & -2 & -3 & 2 \end{bmatrix} \longrightarrow \begin{array}{cccc} b & b & b & \\ \end{array} B = \begin{bmatrix} 4 & -1 & 3 & 1 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

✓ E.F.

B is an E.F of A and has no row of the form $[0\ 0 \cdots 0\ b]$ with $b \neq 0$, so the LES is consistent. To find the soln, we need the R.E.F. ......

Ex: find the unique soln.

2. Parametric Vector Forms.

Vectors.

just like numbers, but multiple numbers at a time.

- Def: A <u>Vector</u> is a list of numbers arranged in a column. e.g. $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

- <u>Operations on Vectors</u>:

(addition). We can add two vectors of the same size, entry-wise.

eg $$\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3+4 \\ 2-1 \\ 1+0 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \\ 1 \end{bmatrix}.$$

(scalar mult.) We can scale a vector by a number:

$$c \cdot \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} ca_1 \\ ca_2 \\ \vdots \\ ca_n \end{bmatrix} \qquad \text{e.g. } 5 \cdot \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 15 \\ 10 \\ 5 \end{bmatrix}.$$

Def. Given vectors $\vec{v}_1, \ldots, \vec{v}_k$ of the same size and numbers $c_1, c_2, \ldots, c_k$, a vector of the form

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$

is called a linear combination of $\vec{v}_1, \ldots, \vec{v}_k$.

e.g.

$$3 \cdot \begin{bmatrix} 5 \\ 1 \end{bmatrix} - 2 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \cdot 5 - 2 \cdot 1 \\ 3 \cdot 1 - 2 \cdot 0 \end{bmatrix} = \begin{bmatrix} 13 \\ 3 \end{bmatrix}.$$

So $\begin{bmatrix} 13 \\ 3 \end{bmatrix}$ is a lin. comb of $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Next time:
new language for LES. in terms of vectors and matrices.