# Math 2130. Lecture 39.

04. 21. 2021.

**Last time:**

- Complex numbers and diagonalization.
- Inner product in $\mathbb{R}^n$ ( $\langle , \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ )
  - def. and properties. (e.g. $\langle u, u \rangle = u_1^2 + \cdots + u_n^2 \geq 0$, where "=" holds iff $u = 0$ ) $u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$
  - from inner prod. to length ( $\text{len}(u) = \sqrt{\langle u, u \rangle}$. )
  - from length to distance ( $\text{dist}(u, v) = \text{len}(u - v) = \text{len}(v - u)$. )

**Today:**

- More geometry via inner product
  - normalization
  - orthogonality

# 1. Normalization

(*): property of length $\|cv\| = c\|v\|$.
$c = \frac{1}{\|v\|}$.

**Def:** A <u>unit vector</u> in $\mathbb{R}^n$ is a vector $u \in \mathbb{R}^n$ with $\|u\| = 1$.

**Note:** Given any nonzero vector $0 \neq v \in \mathbb{R}^n$, there is a unique vector $v'$ that has the same direction as $v$ (ie., is a positive multiple of $v$) and has length 1.

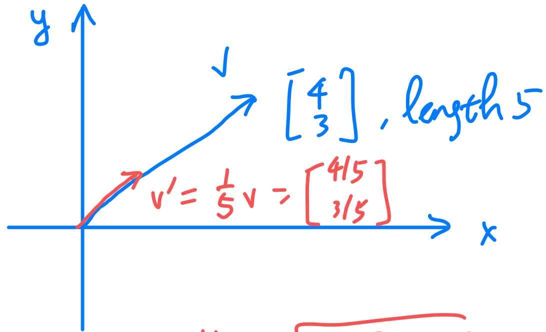
$v' = \lambda v, \lambda > 0$

**Def:** In the above setting, we call $v'$ the <u>normalization</u> of $v$.

**Prop:** The normalization of $0 \neq v \in \mathbb{R}^n$ is given by $v' = \frac{v}{\|v\|}$.

**pf:** Note that $v'$ is of the form $\lambda v$ for $\lambda = \frac{1}{\|v\|} > 0$ and $\|v'\| = \left\| \frac{v}{\|v\|} \right\| \overset{*}{=} \frac{\|v\|}{\|v\|} = 1$. $\square$

e.g.




$$\|v'\| = \sqrt{\left(\tfrac{4}{5}\right)^2 + \left(\tfrac{3}{5}\right)^2} = \sqrt{1} = 1.$$

In the example, we checked that

$$v' = \frac{v}{\|v\|} = \frac{v}{5}$$

does have length 1, so it's the normalization of $v$.

Ex. Normalize $v = \begin{bmatrix}2\\3\\7\end{bmatrix} \in \mathbb{R}^3$.

Soln: The normalization of $v$ should be

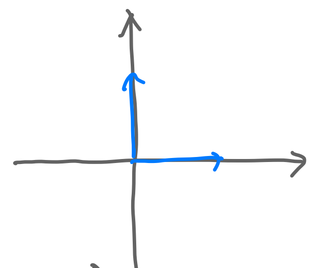
$$v' = \frac{v}{\|v\|} = \frac{1}{\sqrt{2^2+3^2+7^2}} \cdot \begin{bmatrix}2\\3\\7\end{bmatrix} = \frac{1}{\sqrt{62}}\begin{bmatrix}2\\3\\7\end{bmatrix} = \begin{bmatrix}2/\sqrt{62}\\3/\sqrt{62}\\7/\sqrt{62}\end{bmatrix} = \begin{bmatrix}\sqrt{62}/31\\3\sqrt{62}/62\\7\sqrt{62}/62\end{bmatrix}.$$

## 2. Orthogonality.

**Def:** We say that two vectors $u, v \in \mathbb{R}^n$ are $\overbrace{\underline{\text{orthogonal}} \text{ or } \underline{\text{perpendicular}}}^{\text{geometric notion}}$ to each other, and we write $u \perp v$, if $\underbrace{u \cdot v = 0}_{\text{algebraic formulation}}$.

e.g. $n = 2$.

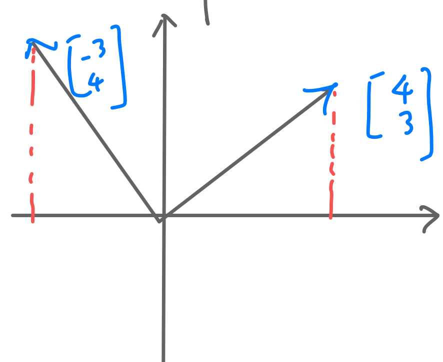
(1) $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.



→ seen to be orth.

$e_1 \cdot e_2 = 1 \cdot 0 + 0 \cdot 1 = 0$ ↑ matches geom. intuition

(2) $u = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$, $v = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

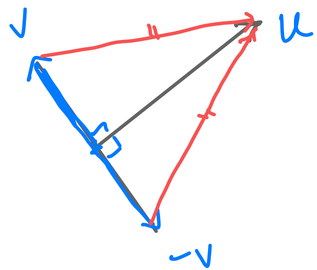


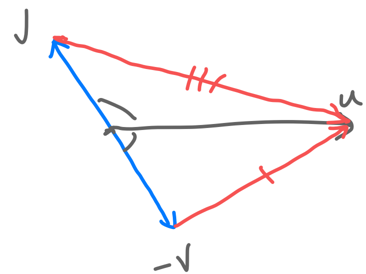

→ seen to be orthog.

$u \cdot v = 4 \cdot (-3) + 3 \cdot 4 = 0$. ✓

An explanation for why we say $u \perp v \iff u \cdot v = 0$ : (combine (1) and (2)).

Note that $u \perp v \overset{(1)}{\iff} \text{dist}(u, v) = \text{dist}(u, -v)$.




But $\text{dist}(u, v) = \text{dist}(u, -v)$

(2).

$\iff \| u - v \|^2 = \| u + v \|^2$

$\iff \langle u-v, u-v \rangle = \langle u+v, u+v \rangle$

$\iff \cancel{\langle u, u \rangle} - 2\langle u, v \rangle + \cancel{\langle v, v \rangle} = \cancel{\langle u, u \rangle} + 2\langle u, v \rangle + \cancel{\langle v, v \rangle}$

$\langle u, v \rangle = 0$.

$\Updownarrow$

**Def:** A set of vectors $\{v_1, v_2, \ldots, v_k\} \subseteq \mathbb{R}^n$ is called an orthogonal set if they are pairwise orthogonal, i.e., if $v_i \cdot v_j = 0$ whenever $i \neq j$.

**E.g.** $S = \left\{ \underset{u_1}{\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}}, \underset{u_2}{\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}}, \underset{u_3}{\begin{bmatrix} -\frac{1}{2} \\ -2 \\ \frac{7}{2} \end{bmatrix}} \right\}$. Show that $S$ is an ortho. set.

**Pf:** Need $u_1 \perp u_2$, $u_1 \perp u_3$ and $u_2 \perp u_3$.

"$u_1 \perp u_2$": $u_1 \cdot u_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = -3 + 2 + 1 = 0$, so $u_1 \perp u_2$.

"$u_1 \perp u_3$": $u_1 \cdot u_3 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -\frac{1}{2} \\ -2 \\ \frac{7}{2} \end{bmatrix} = -\frac{3}{2} - 2 + \frac{7}{2} = 0$, so $u_1 \perp u_3$.

"$u_2 \perp u_3$": $u_2 \cdot u_3 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -\frac{1}{2} \\ -2 \\ 7/2 \end{bmatrix} = \frac{1}{2} - 4 + \frac{7}{2} = 0$, so $u_2 \perp u_3$. $\square$

Q: Why are orthogonal sets interesting? $\begin{bmatrix}1\\1\end{bmatrix} \begin{bmatrix}1\\2\end{bmatrix}$.

One reason: they are automatically linearly independent.

Thm: If $S = \{v_1, \cdots, v_k\}$ is an orthogonal set of nonzero elts, then $S$ is lin. ind.

Eg. The theorem can allow us that the set $S$ from the previous example is lin ind. without computing any echelon forms.

Pf. Suppose $S$ is orthogonal, and suppose $a_1 V_1 + a_2 V_2 + \cdots + a_k V_k = 0$, need $a_1 = a_2 = \cdots = a_k = 0$.

then for each $1 \le i \le k$, $\langle V_i, a_1 V_1 + a_2 V_2 + \cdots + a_k V_k \rangle = \langle V_i, 0 \rangle = 0$.

Thus, $a_1 \langle V_i, V_1 \rangle + a_2 \langle V_i, V_2 \rangle + \cdots + a_k \langle V_i, V_k \rangle = 0$,

so $a_i \underbrace{\langle V_i, V_i \rangle}_{\neq 0} = 0 \Rightarrow a_i = 0$. But $i$ is arbitrary.

so $a_1 = a_2 = \cdots = a_k = 0$. ✓. ∎

**Ex.** Show that if $u \perp v$, $u \perp w$ for some $u, v, w \in \mathbb{R}^n$, then

$u \perp x$ for all $x \in \text{span}\{v, w\}$.

**pf:** By the def. of span, $x$ must be of the form

$$x = av + bw \quad \text{for some } a, b \in \mathbb{R}.$$

But then $\langle u, x \rangle = \langle u, av + bw \rangle = \langle u, av \rangle + \langle u, bw \rangle$

$$= a \underline{\langle u, v \rangle} + b \underline{\langle u, w \rangle}$$

(since $u \perp v$, $u \perp w$)

$$= a \cdot 0 + b \cdot 0 = 0.$$

It follows that $u \perp x$. $\square$

next time: projections.

orthonormal sets.