Math 2130. Lecture 38.

Final Exam: available 11:59am – 11:59pm. May 5th.

04. 19. 2021.

Last time: · diagonalization and matrices of linear maps: $\left(\underset{n\times n}{A} \to T_A : \mathbb{R}^n \to \mathbb{R}^n, \; x \mapsto Ax \right)$

to diagonalize an $n\times n$ matrix A

is the same as to find a basis B of $\mathbb{R}^n$ s.t. $[T_A]_B^B$ is diagonal.

Today: · finish Ch. 5: complex eigenvalues

· start Ch. 6: 6.1: inner product, length and distance.

1. Complex eigenvalues.

Consider the polynomial $f(x) = x^2+1$. It has no soln over the real numbers.

$(\mathbb{R})$

However, it has a pair of solns over the complex numbers $\mathbb{C}$.

Here, · a complex number is a number of the form $a+bi$ where $a, b \in \mathbb{R}$ and i is a number s.t. $i^2 = -1$.

· Addition and multiplication works in a natural way:

$$(a+bi) + (a'+b'i) = (a+a') + (b+b')i$$

$$(a+bi)(c+di) = ac + bci + adi + bd\underbrace{i^2}_{-1}$$

$$= (ac-bd) + (bc+ad)i.$$

· <u>Fact</u>: given $n \geq 1$, every poly. f. of deg n factors into n linear factors. In other words, f has n complex roots.

e.g. $f(x) = x^2+1 \longrightarrow$ roots: i, $-i$, $x^2+1 = (x-i)(x+i)$

$g(x) = x^2+6x+10 \longrightarrow \Delta = b^2-4ac = 6^2-4.10 = -4 < 0$. no real root. complex roots

complex roots

$$\frac{-6 \pm \sqrt{-4}}{2} = \frac{-6 \pm 2i}{2}$$

$$= -3 \pm i$$

Point: · Over the real numbers $\mathbb{R}$, a polynomial of deg. n may not have n real roots; it may have no root at all.

· However, over the complex numbers $\mathbb{C}$, every poly. of deg n has n roots, so every $n\times n$ matrix has eigenvalues.

Eg. Working over the complex numbers, diagonalize $A = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}$.

Soln: $\text{char}_A(x) = \begin{vmatrix} 1-x & -2 \\ 2 & 3-x \end{vmatrix} = (1-x)(3-x)+4 = x^2-4x+3+4 = x^2-4x+7$.

the evalues: $\dfrac{4 \pm \sqrt{-12}}{2\cdot 1} = \dfrac{4 \pm 2\sqrt{3}\cdot i}{2} = 2 \pm \sqrt{3}\, i$.

$\lambda_1 = 2+\sqrt{3}\, i$, $E_{\lambda_1} = \text{Null}\left(\begin{bmatrix} 1-2-\sqrt{3}i & -2 \\ 2 & 3-2-\sqrt{3}i \end{bmatrix}\right)$

$\begin{bmatrix} -1-\sqrt{3}i & -2 & 0 \\ 2 & 1-3\sqrt{3}i & 0 \end{bmatrix} \longrightarrow \left\{ \begin{bmatrix} -2 \\ 1+\sqrt{3}i \end{bmatrix} \right\}$ is a basis of E_{λ_1}. ...

Similarly, can find a basis for E_{λ_2} and finish the diagonalization.

2. Inner products, length, and distance. (6.1)

We return to the vec. spaces of the form $\mathbb{R}^n$.

Def. (inner products) The inner product of two elts $u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$

is the scalar $u \cdot v = \langle u, v \rangle := u^T \cdot v$

ie, $\langle u, v \rangle = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$

E.g. $n=2$, $\left\langle \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\rangle = 1 \cdot 3 + 2 \cdot 4 = 11$

$n=3$, $\left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle = 1 \cdot 0 + 0 \cdot 0 + 0 \cdot 1 = 0.$

Thm. (properties of inner products) Let $u, v, w \in \mathbb{R}^n$, $c \in \mathbb{R}$. Then

(1). $u \cdot v = v \cdot u$

(2). $(u+v) \cdot w = u \cdot w + v \cdot w$

(3). $c(u \cdot v) = u \cdot (cv) = (cu) \cdot v$

(4). $u \cdot u \geq 0 \ \forall u \in \mathbb{R}^n$, and $u \cdot u = 0$ iff $u = \vec{0}$.

Pf: (1) Say $u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$, $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$, then $u \cdot v = u_1 v_1 + \cdots + u_n v_n$

$= v_1 u_1 + \cdots + v_n u_n = v \cdot u$.

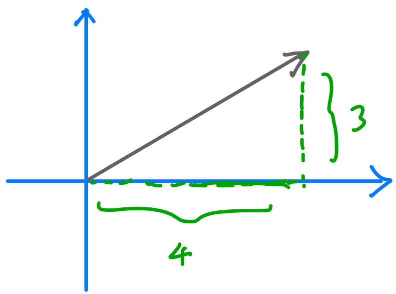
(2). (3): also routine $\rightarrow$ Ex.

(4). $u \cdot u = u_1^2 + \cdots + u_n^2 \geq 0$, $u \cdot u = 0 \iff u_i^2 = 0 \ \forall i \iff u_i = 0 \ \forall i$

$\iff u = \vec{0}$.

inner products → length

E.g.



$n=2$, $v=\begin{bmatrix}4\\3\end{bmatrix}$

By the Pythagorean theorem, we'd say the length of $\vec{v}$ is $\sqrt{4^2+3^2}=5$

Note: $4^2+3^2=\begin{bmatrix}4\\3\end{bmatrix}\cdot\begin{bmatrix}4\\3\end{bmatrix}=u\cdot u$.

generalize

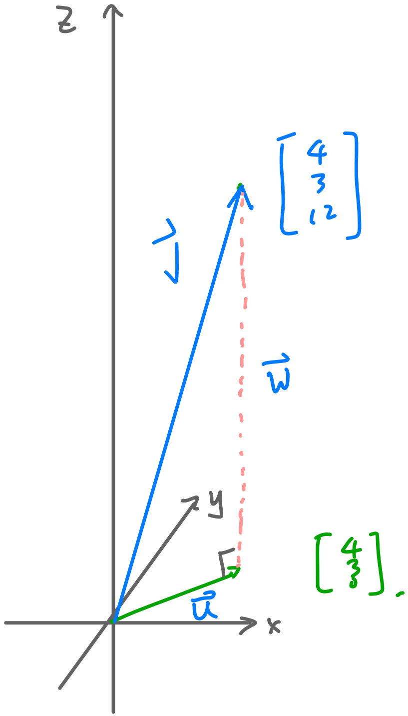
Def. We define the length of a vector $v=\begin{bmatrix}v_1\\ \vdots\\ v_n\end{bmatrix}\in\mathbb{R}^n$ to be

any $n\geq 1$.

$$\|v\|:=\sqrt{v\cdot v}=\sqrt{v_1^2+v_2^2+\cdots+v_n^2}$$

Rmk: Our examples shows that the above definition of length agrees with our intuition from Euclidean geometry.

E.g. $n=3$, $v = \begin{bmatrix} 4 \\ 3 \\ 12 \end{bmatrix} \in \mathbb{R}^3$.



By our def, $\|v\| = \sqrt{4^2 + 3^2 + 12^2} = 13$.

Also, $\|v\| = \sqrt{\|u\|^2 + \|w\|^2}$

$= \sqrt{(4^2+3^2) + 12^2} = 13$.

(Thus, in $\mathbb{R}^3$, the def. of $\|v\|$ also captures Euclidean geometry.)

length → distance

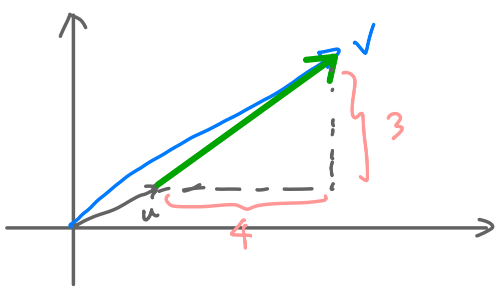
next time: orthogonality via inner products.

Def: Let $u, v \in \mathbb{R}^n$. We define the <u>distance from v to u</u> to be

$$\text{dist}(v, u) = \| u - v \| .$$

Note: Clearly $\|u-v\| = \|v-u\|$ since $\|w\| = \|-w\|$, so $\text{dist}(v,u) = \text{dist}(u,v)$ and they may both just be called the distance between u and v.

Eg. $n = 2$, $u = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. $v = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$. $\text{dist}(u,v) = \|v - u\|$



$$= \left\| \begin{bmatrix} 6 \\ 4 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\|$$

$$= \left\| \begin{bmatrix} 4 \\ 3 \end{bmatrix} \right\| = \sqrt{4^2 + 3^2} = 5.$$