Math 2130. Lecture 37.

04. 16. 2021.

Last time: · more diagonalization examples.

· why $\overset{n\times n}{A} = PDP^{-1}$: consider the map $T_A : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto Ax$

and its matrices w.r.t. the standard basis

and the eigenbasis.

Today: examples, hw. discussions.

Recall: Given a linear map $T: V \to W$, a basis β of V $\{b_1, \ldots, b_k\}$

and a basis γ of W, the matrix of T

w.r.t. β and γ is

$$[T]_\beta^\gamma = \Big[\, [T(b_1)]_\gamma \ \cdots \ [T(b_k)]_\gamma \,\Big]$$

Def: If $V = W$ and $\beta = \gamma$, we write $[T]_\beta = [T]_\beta^\beta$ and call it the β-matrix of T.

E.g. ii) $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$ (Lecture 35)

C : the standard basis

$\lambda_1 = -7 \rightarrow B_1 = \left\{ \underbrace{\begin{bmatrix} -1 \\ 3 \end{bmatrix}}_{v_1} \right\}$

$\lambda_2 = 3 \rightarrow B_2 = \left\{ \underbrace{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}_{v_2} \right\}$

$\rightarrow B = \left\{ \underset{[v_1]_C}{\begin{bmatrix} -1 \\ 3 \end{bmatrix}}, \underset{[v_2]_C}{\begin{bmatrix} 3 \\ 1 \end{bmatrix}} \right\}$ is an eigenbasis for A

$\rightarrow P = \begin{bmatrix} -1 & 3 \\ 3 & 1 \end{bmatrix} = \underset{C \leftarrow B}{P}$

$T_A : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad v \longmapsto Av.$

$T(v_1) = -7v_1, \quad T(v_2) = 3v_2$, so

$[T]_B^B = \begin{bmatrix} [T(v_1)]_B & [T(v_2)]_B \end{bmatrix}$

$= \begin{bmatrix} [-7v_1]_B & [3v_2]_B \end{bmatrix}$

$= \begin{bmatrix} -7 & 0 \\ 0 & 3 \end{bmatrix} = D.$

$A = PDP^{-1}$

$\Updownarrow$

$[T]_C^C = \underset{C \leftarrow B}{P} \, [T]_B^B \, \underset{B \leftarrow C}{P}.$

(2). $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$ (Lec. 35.)

C : the standard basis of $\mathbb{R}^3$.

$$\lambda_1 = 1 \longrightarrow B_1 = \left\{ \underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}_{v_1} \right\}$$

$$\lambda_2 = -2 \longrightarrow B_2 = \left\{ \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}_{v_2}, \underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}_{v_3} \right\}$$

$$\longrightarrow B = \{v_1, v_2, v_3\} \rightarrow P = [v_1\, v_2\, v_3] = \underset{C \leftarrow B}{P}$$

$$T_A : \mathbb{R}^3 \longrightarrow \mathbb{R}^3, \quad v \longmapsto Av$$

$$T_A(v_1) = 1 \cdot v_1, \quad T_A(v_2) = -2v_2, \quad T_A(v_3) = -2v_3 \Longrightarrow [T]_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = D$$

$$A = PDP^{-1} \Longleftrightarrow [T_A]_C^C = \underset{C \leftarrow B}{P} \, [T]_B^B \, \underset{B \leftarrow C}{P} .$$

Upshot: To find a basis B s.t. $[T_A]_B^B$ is diagonal is the same as to find a diagonalization of the matrix A.

E.g. Ex 5.4.13. Consider $T: \mathbb{R}^2 \to \mathbb{R}^2$, $v \longmapsto Av$ where

$$A = \begin{bmatrix} 0 & 1 \\ -3 & 4 \end{bmatrix}.$$

Find a basis B of $\mathbb{R}^2$ s.t. $[T]_B = [T]_B^B$ is diagonal.

Soln: $\mathrm{char}_A(x) = \begin{vmatrix} 0-x & 1 \\ -3 & 4-x \end{vmatrix} = -x(4-x) + 3 = x^2 - 4x + 3$

$= (x-1)(x-3)$

eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 3$, both w/ alg. mult $= 1$.

In particular, A is diagonalizable.

$$E_{\lambda_1} = \mathrm{Null}\,(A - \lambda_1 I) = \mathrm{Null}\begin{bmatrix} 0-1 & 1 \\ -3 & 4-1 \end{bmatrix} = \mathrm{Null}\left(\begin{bmatrix} -1 & 1 \\ -3 & 3 \end{bmatrix}\right).$$

Since $\dim E_{\lambda_1} = 1$ and $v_1 := \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in E_{\lambda_1}$, $B_1 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is a basis of E_{λ_1}.

$$E_{\lambda_2} = \mathrm{Null}\,(A - \lambda_2 I) = \mathrm{Null}\left(\begin{bmatrix} 0-3 & 1 \\ -3 & 4-3 \end{bmatrix}\right) = \mathrm{Null}\left(\begin{bmatrix} -3 & 1 \\ -3 & 1 \end{bmatrix}\right),$$

Since $\dim E_{\lambda_2} = 1$ and $v_2 := \begin{bmatrix} 1 \\ 3 \end{bmatrix} \in E_{\lambda_2}$, $B_2 = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$ is a basis of E_{λ_2}.

It follows that for the basis $B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$ is a basis of $\mathbb{R}^2$

s.t. $[T]_B$ is diagonal (in fact, $[T]_B = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$). $\square$

Ex 5.13.14 (Hw.)

$$A = \begin{bmatrix} 5 & -3 \\ -7 & 1 \end{bmatrix}, \quad T: \mathbb{R}^2 \to \mathbb{R}^2, \ v \mapsto Av \quad \forall v \in \mathbb{R}^2.$$

Todo: find a basis B of $\mathbb{R}^2$ s.t. $[T]_B$ is diagonal.

- mimic the last problem: diagonalize A.

Ex. 5.4.17. Let $A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$ and $B = \{b_1, b_2\}$ for

$b_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $b_2 = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$. Define $T: \mathbb{R}^2 \to \mathbb{R}^2$, $x \mapsto Ax$.

(a). Verify that b_1 is an e-vec but A is not diag.

(b). Find the B-matrix of T.

Soln: (a).
$$A b_1 = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2 b_1,$$

so b_1 is an evec. with evalue 2.

$$\mathrm{char}_A(x) = \begin{vmatrix} 1-x & 1 \\ -1 & 3-x \end{vmatrix} = (1-x)(3-x) + 1 = x^2 - 4x + 4 = (x-2)^2$$

So A has a unique evalue $\lambda = 2$ of alg. mult 2.

$$E_\lambda = \text{Null}\left(\begin{bmatrix} 1-2 & 1 \\ -1 & 3-2 \end{bmatrix}\right) = \text{Null}\left(\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\right)$$

$$\begin{bmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\to E_\lambda = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x-y=0,\ y \in \mathbb{R} \right\} = \left\{ y \begin{bmatrix} 1 \\ 1 \end{bmatrix} : y \in \mathbb{R} \right\}.$$

So a basis of E_λ is $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ and $\text{geom. mult}(\lambda) = \dim E_\lambda = 1$.

Since geom. mult $(\lambda) <$ alg. mult (λ), A is not diagalizable.

(b). (by (a), we shouldn't expect $[T]_B$ to be diagonal.)

$T(b_1) = 2b_1$, so $[T(b_1)]_B = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$,

$$T(b_2) = Ab_2 = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \end{bmatrix} = ?\underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{b_1} + ??\underbrace{\begin{bmatrix} 5 \\ 4 \end{bmatrix}}_{b_2}.$$

$$\begin{bmatrix} 1 & 5 & 9 \\ 1 & 4 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & 9 \\ 0 & -1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & 9 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

So $T(b_2) = -1\, b_1 + 2b_2$, hence $[T(b_2)]_B = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

It follows that

$$[T]_B = \left[[T(b_1)]_B \;\middle|\; [T(b_2)]_B \right] = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}. \qquad \square$$

indeed not diag, as expected!