Math 2130. Lecture 36.

04. 14. 2021.

Last time: · what diagonalizable matrices are: square matrices similar to diagonal matrices: A s.t. $\exists\, D = \mathrm{diag}(a_1, \ldots, a_n)$ and P inv w/ $A = PDP^{-1}$.

· why diagonalizable matrices are interesting / nice:

if A is diagonalizable, say $A = PDP^{-1}$, then $A^k = PD^kP^{-1}$.

· when a matrix is diagonalizable: A is diagonalizable iff the geom. mult agrees w/ the alg. mult. for every e-value of A.

if so. how we can diagonalize A

$$A = PDP^{-1} \text{ where } D = \begin{bmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_1 & & & \\ & & & \ddots & & \\ & & & & \lambda_k & \\ & & & & & \ddots \\ & & & & & & \lambda_k \end{bmatrix} \text{ and } P = \Big[\,\underbrace{v_{11} \cdots v_{1n_1}}_{\text{basis for } E_{\lambda_1}} \Big| \cdots \Big| v_{k1} \cdots v_{kn_k}\,\Big]$$

Today: · more practice problems · why $A = PDP^{-1}$ $\rightarrow$ Connection to matrices of linear maps

1. More diag. exercises.

Determine diagonalizability. Diagonalize if possible.

(1). $A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$.

Soln: (sketch)

$$\mathrm{char}_A(x) = \det \begin{bmatrix} 2-x & 4 & 3 \\ -4 & -6-x & -3 \\ 3 & 3 & 1-x \end{bmatrix} \overset{\text{cofactor expansion}}{=} \cdots = -(x-1)(x+2)^2$$

So the eigenvalues of A are $\lambda_1 = 1$ with alg. mult. 1,

and $\lambda_2 = -2$ with alg. mult. 2.

Since alg. mult $(\lambda_1) = 1$, we must have geom. mult $(\lambda_1) = 1 =$ alg. mult (λ_1).

Next we look for geom. mult (λ_2):

$$E_{\lambda_2} = \mathrm{Null}(A - \lambda_2 I_3) = \mathrm{Null}\left(\begin{bmatrix} 2+2 & 4 & 3 \\ -4 & -6+2 & -3 \\ 3 & 3 & 1+2 \end{bmatrix}\right) = \mathrm{Null}\begin{bmatrix} 4 & 4 & 3 \\ -4 & -4 & -3 \\ 3 & 3 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 4 & 3 \\ -4 & -4 & -3 \\ 3 & 3 & 3 \end{bmatrix} \to \begin{bmatrix} 4 & 4 & 3 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 4 & 4 & 3 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{array}{c} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ p & f & p \end{array} \end{array} \searrow \text{E.F.}$$

Thus, $\text{geom. mult}(\lambda_2) = \dim E_{\lambda_2} = \#$ non-pivot cols in A

$$= 1.$$

Since $\text{geo. mult}(\lambda_2) = 1 < 2 = \text{alg. mult}(\lambda_2)$,

the matrix A is not diagonalizable.

(2). $A = \begin{bmatrix} -2 & 12 \\ -1 & 5 \end{bmatrix}$

Soln: $\mathrm{char}_A(x) = \begin{vmatrix} -2-x & 12 \\ -1 & 5-x \end{vmatrix} = (-2-x)(5-x) + 12 = x^2 - 3x - 10 + 12$

$$= x^2 - 3x + 2 = (x-1)(x-2)$$

So the evalues of A are $\lambda_1 = 1$ and $\lambda_2 = 2$, both with alg. mult. 1.

Since both the alg. mult. are 1, A must be diagonalizable.

E_{λ_1}: $E_{\lambda_1} = \mathrm{Null}\left(\begin{bmatrix} -2-1 & 12 \\ -1 & 5-1 \end{bmatrix}\right) = \mathrm{Null}\left(\begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix}\right).$

Note that we should have $\dim E_{\lambda_1} = \text{geom. mult}(\lambda_1) = 1$.

and that $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ is clearly in E_{λ_1}.

Therefore $\left\{\begin{bmatrix} 4 \\ 1 \end{bmatrix}\right\}$ must be a basis of E_{λ_1}.

$\underline{E_{\lambda_2}}$: $E_{\lambda_2} = \text{Null}\left(\begin{bmatrix} -2-2 & 12 \\ -1 & 5-2 \end{bmatrix}\right) = \text{Null}\left(\begin{bmatrix} -4 & 12 \\ -1 & 3 \end{bmatrix}\right)$.

Note that $\dim E_{\lambda_2} = 1$ as in the λ_1-case

and $\begin{bmatrix} 3 \\ 1 \end{bmatrix} \in E_{\lambda_2}$,

So $\left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$ is a basis of E_{λ_2}.

It follows that $A = PDP^{-1}$ where

$$D = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 2 \end{array}\right] \text{ and } P = \left[\begin{array}{c|c} 4 & 3 \\ 1 & 1 \end{array}\right]. \qquad \square$$

Note: If you don't need it for further computations, you don't have to find P^{-1}.

(13). $A = \begin{bmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & 2 \end{bmatrix}$.

Soln: $\operatorname{char}_A(x) = \det \begin{bmatrix} 5-x & -8 & 1 \\ 0 & 0-x & 7 \\ 0 & 0 & 2-x \end{bmatrix} \overset{\text{Ch. 3.}}{=} (5-x)(0-x)(2-x) = -x^1(x-2)^1(x-5)^1$

(Indeed, given a triangular mat. $B = \begin{bmatrix} a_1 & & * \\ & a_2 & \\ 0 & & \ddots \; a_n \end{bmatrix}$, $\operatorname{char}_B(x) = (a_1-x)(a_2-x)\cdots(a_n-x)$

and hence the evalues are just the diag. entries $a_1, a_2, \ldots, a_n$ in B.)

Thus, the eigenvalues of A are 0, 2, and 5, all with alg. mult 1.

In particular, A is diagonalizable.

Diagonalization (finding P & D s.t. $A = PDP^{-1}$.) : ... Ex.

2. Why $A = PDP^{-1}$.

Setting: Let A be a diagonalizable $n \times n$ matrix. $\lambda_1, \cdots, \lambda_k$ are the distinct evalues of A with alg. mult ($=$ geom. mult) $n_1, \cdots, n_k$; for each $1 \leq i \leq k$, E_{λ_i} has basis $B_i = \{v_{i1}, v_{i2}, \cdots, v_{in_i}\}$. Finally, write $[B_i] := [v_{i1} | v_{i2} | \cdots | v_{in_i}]$

Claim: $A = PDP^{-1}$ where $P = [B_1 | B_2 | \cdots | B_k]$ and $D = \text{diag}(\underbrace{\lambda_1, \cdots, \lambda_1}_{n_1}, \cdots, \underbrace{\lambda_k, \cdots, \lambda_k}_{n_k})$.

Consider the map $T_A: \mathbb{R}^n \to \mathbb{R}^n$ given by

$$T_A(x) = A \cdot x \quad \forall x \in \mathbb{R}^n.$$

Recall that for a linear map $T: V \to W$, a basis β $= \{b_1, \cdots, b_n\}$ of V, and a basis γ of W, then the matrix of T relative to β and γ is ...

... the matrix $[T]_\beta^\gamma = \left[[T(b_1)]_\gamma \quad \cdots \quad \cdots \quad [T(b_n)]_\gamma \right].$

In particular, for the standard basis $C = \{ e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, e_n \}$ of $\mathbb{R}^n$, we have

$$[T_A]_C^C = \left[[T(e_1)]_C \cdots [T(e_n)]_C \right] = A$$

and for $B = B_1 \cup B_2 \cdots \cup B_k$,

$$[T_A]_B^B = \left[[T_A(v_{11})]_B \; [T_A(v_{12})]_B \cdots [T_A(v_{1n_1})]_B \;\middle|\; \cdots \;\middle|\; [T(v_{k1})]_B \cdots [T(v_{kn_k})]_B \right]$$

$x \mapsto Ax$

$$= \left[[Av_{11}]_B \;\middle|\; \cdots \; [Av_{1n_1}]_B \;\middle|\; \cdots \;\middle|\; [Av_{k1}]_B \cdots [Av_{kn_k}]_B \right]$$

$$= \left[[\lambda_1 v_{11}]_B \cdots [\lambda_1 v_{1n_1}]_B \;\middle|\; \cdots \quad \cdots \right]$$

$$= \mathrm{diag}(\underbrace{\lambda_1, \lambda_1, \ldots, \lambda_1}_{n_1}, \underbrace{\lambda_2, \ldots, \lambda_2}_{n_2}, \ldots) = D \,!$$

What have we done?

We have $\underset{B,C}{T_A : \mathbb{R}^n} \to \underset{B,C}{\mathbb{R}^n}$ and two bases C and B of $\mathbb{R}^n$.

We noted that $[T_A]_C^C = A$ and $[T_A]_B^B = D$.

Recall that $\underset{C \leftarrow B}{P} = \left[[v_{11}]_C \; \cdots \; [v_{k n_k}]_C \right] = \left[v_{11} \; \cdots \; v_{k n_k} \right] = P$ (since C is standard)

Thus, the equation $A = PDP^{-1}$ can be interpreted as

$$[T_A]_C^C = \underset{C \leftarrow B}{P} \, [T_A]_B^B \left(\underset{C \leftarrow B}{P} \right)^{-1}$$

or $[T_A]_C^C = \underset{C \leftarrow B}{P} \, [T_A]_B^B \, \underset{B \leftarrow C}{P}$. $\longrightarrow$ $\left\{ \begin{array}{l} \text{true, since } \forall v \in \mathbb{R}^n \text{ we have} \\ [T_A]_C^C \cdot v = \underset{C \leftarrow B}{P} \cdot [T_A]_B^B \cdot \underset{B \leftarrow C}{P} \cdot v \end{array} \right.$

The equation $A = PDP^{-1}$ holds since the last equation holds.