Math 2130. Lecture 35.

04.12.2021.

Last time:

Let A be a $n \times n$ matrix.

- finding eigenvectors / bases of eigenspaces $E_\lambda = \{ v \mid Av = \lambda v \}$
- algebraic mult. vs. geometric mult ($\dim E_\lambda$)
- **Thm:** For every e-value λ of A, we have $1 \leq \text{geom.mult}(\lambda) \leq \text{alg.mult.}(\lambda)$; if $\text{geom.mult}(\mu) = \text{alg.mult}(\mu)$ for all e-value μ of A, then the union of the bases of the espaces is a basis of $\mathbb{R}^n$. → "nice": means that A is diagonalizable

Today:

- diagonalizable matrices: def and applications
- diagonalization: algorithm, connection to eigenvectors / evalues, examples.

1. Diagonalizable matrices.

Def. (similarity of matrices). Two $n \times n$ matrices A, B are called <u>similar</u> if there is an invertible matrix P s.t. $B \overset{*}{=} PAP^{-1}$; in this case we write $A \sim B$.

$$\Updownarrow$$

$$\left(\underset{Q}{\underline{\underline{P^{-1}}}} B \underset{Q^{-1}}{\underline{\underline{P}}} = A \right)$$

Note: If A is similar to B, then

$$\det(A) = \det(B) \; : \quad \det B \overset{*}{=} \det(PAP^{-1}) = \det P \det A \det(P^{-1})$$

$$= \det A \det P \, (\det P)^{-1}$$

$$= \det A.$$

Def. (diagonalizability) We say an $n \times n$ matrix A is diagonalizable if it is similar to a diagonal matrix, i.e., if there is diag. matrix D and an invertible mat. P s.t. $A = PDP^{-1}$.

Ex: $A \sim B \Rightarrow \mathrm{char}_A(x) = \mathrm{char}_B(x)$

Why are diagonalizable matrices interesting?

Main reason for us: their powers are easier to compute than those of general matrices: if A is diagonalizable, say

$A = PDP^{-1}$ for a diagonal matrix

$$D = \mathrm{diag}(a_1, a_2, \to a_n) = \begin{bmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_n \end{bmatrix},$$

then

$$A^k = \underbrace{PD\cancel{P^{-1}} \cdot \cancel{P}D\cancel{P^{-1}} \cdot \cancel{P}D\cancel{P^{-1}} \cdot \cancel{\ } \ldots}_{k \text{ copies}} \cancel{\ } \cdot \cancel{P}DP^{-1}$$

$$= P \cdot D^k \cdot P^{-1} = P \underbrace{\mathrm{diag}(a_1^k, a_2^k, \ldots, a_n^k)}_{\text{relatively easy to compute.}} P^{-1}.$$

E.g. $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$ from the last lecture.

Recall that A has evalues $\lambda_1 = -7$ and $\lambda_2 = 3$,

with $\text{alg.mult}(\lambda_i) = \text{geom.mult}(\lambda_i) = 1$ $\forall i \in \{1, 2\}$.

Also recall that E_{λ_1} has basis $\{ \underbrace{\begin{bmatrix} -1 \\ 3 \end{bmatrix}}_{v_1} \}$ E_{λ_2} has basis $\{ \underbrace{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}_{v_2} \}$.

It will turn out that we have $A = PDP^{-1}$ where $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -7 & 0 \\ 0 & 3 \end{bmatrix}$

and $P = [v_1 | v_2] = \begin{bmatrix} -1 & 3 \\ 3 & 1 \end{bmatrix}$. Thus,

$$A^k = PD^kP^{-1} = \begin{bmatrix} -1 & 3 \\ 3 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} -7^k & 0 \\ 0 & 3^k \end{bmatrix}}_{\text{at most two hard computations}} \checkmark \begin{bmatrix} \frac{2}{7} & \frac{1}{7} \\ \frac{1}{7} & -\frac{2}{21} \end{bmatrix}; \longrightarrow \text{easy.}$$

On the other hand, computing $A^4 = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$ is tedious.

2. Diagonalizability and diagonalization via evectors.

Let A be an $n \times n$ matrix.

Thm. · If we have geom. mult (λ) = alg. mult (λ) for every evalue λ of A, then A is diagonalizable. Moreover, if $\lambda_1, \dots, \lambda_k$ are the distinct evalues of A and $B_1 = \{v_{11}, v_{12}, v_{13}, \dots, v_{1n_1}\}$, $B_2 = \{v_{21}, v_{22}, \dots, v_{2n_2}\}, \dots, B_k = \{v_{k1}, \dots, v_{kn_k}\}$ are the bases of $E_{\lambda_1}, \dots, E_{\lambda_k}$, respectively, then $A = PDP^{-1}$ where

$$D = \operatorname{diag}(\underbrace{\lambda_1, \lambda_1, \dots, \lambda_1}_{n_1}, \underbrace{\lambda_2, \dots, \lambda_2}_{n_2}, \dots, \underbrace{\lambda_k, \dots, \lambda_k}_{n_k}) \text{ and } P = \left[v_{11} \,\middle|\, \cdots \,\middle|\, v_{1n_1} \,\middle|\, \cdots \,\middle|\, v_{k1} \,\middle|\, \cdots \,\middle|\, v_{kn_k} \right].$$

· If for some evalue λ of A we have geom. mult (λ) < alg. mult (λ), then A is not diagonalizable, i.e, we cannot find a diagonal matrix D and an inv. matrix P s.t $A = PDP^{-1}$.

Corollary. If alg. mult $(\lambda) = 1$ for all evalue λ of A, then A is diagonalizable.

Examples. (1). $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$. → also from last time

$\lambda_1 = 1$, geom. mult = alg. mult = 1 ; $\lambda_2 = -2$, geom. mult = alg. mult $= 2$.

$\Downarrow$

A is diagonalizable.

$\downarrow$

$B_1 = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$

$\quad v_{11}$

$\downarrow$

$B_2 = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\quad v_{21} \quad v_{22}$

$\Downarrow$

$A = PDP^{-1}$ where $P = \left[\begin{array}{c|cc} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{array}\right]$ and $D = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & -2 & 0 \\ 0 & 0 & -2 \end{array}\right]$.

(2). $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$. $\rightarrow$ $\underbrace{\text{diagonalize}}_{\text{find a } (P,D) \text{ pair w/ } A = PDP^{-1}.}$ it and find A^k.

Soln:

$$\text{char}_A(x) = \begin{vmatrix} 7-x & 2 \\ -4 & 1-x \end{vmatrix} = (7-x)(1-x)+8 = x^2-8x+7+8$$

$$= x^2-8x+15 = (x-3)(x-5)$$

So $\lambda_1 = 3$, $\lambda_2 = 5$ are the eigenvalues, each of alg. mult 1 and hence also of geom. mult. 1. In particular, A must be diagonalizable. Let's find "B_1" and "B_2":

$$E_{\lambda_1} = \text{Nul}(A - \lambda_1 I) = \text{Nul}\left(\begin{bmatrix} 4 & 2 \\ -4 & -2 \end{bmatrix}\right).$$

$$\left.\begin{array}{l} \text{Note that we should have } \dim E_{\lambda_1} = \text{geom. mult}(\lambda_1) = 1 \\ \text{and that } v_{11} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \in E_{\lambda_1} \text{ since } \begin{bmatrix} 4 & 2 \\ -4 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = 0. \end{array}\right\} \Rightarrow B_1 = \left\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right\} \text{ is a basis of } E_{\lambda_1}.$$

$$E_{\lambda_2} = \text{Null}(A - \lambda_2 I) = \text{Null}\left(\begin{bmatrix} 2 & 2 \\ -4 & -4 \end{bmatrix}\right).$$

Similarly, we note that dim E_{λ_2} = geom. mult $(\lambda_2) = 1$.

and that $\begin{bmatrix} 1 \\ -1 \end{bmatrix} \in E_{\lambda_2}$

$$\frac{1}{1}\begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$$

So $B_2 = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ is a basis of E_{λ_2}.

By our theorem, we may diagonalize A as $A = PDP^{-1} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}^{-1}.$

It follows that $A^k = (PDP^{-1})^k = PD^kP^{-1}$

$$= \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}\underbrace{\begin{bmatrix} 3^k & 0 \\ 0 & 5^k \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}}$$

$$= \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} -3^k & -3^k \\ 2\cdot 5^k & 5^k \end{bmatrix} = \begin{bmatrix} -3^k + 2\cdot 5^k & -3^k + 5^k \\ 2\cdot 3^k - 2\cdot 5^k & 2\cdot 3^k - 5^k \end{bmatrix}. \quad \square$$

next time: more examples of diagonalization, why $A = PDP^{-1}$.