Math 2130. Lecture 34.

04. 09. 2021.

Last time: Let A be an $n \times n$ matrix.

- The characteristic poly. of A is $\mathrm{char}_A(x) = \det[A - xI_n]$
- The eigenvalues of A are exactly the roots of $\mathrm{char}_A(x)$; equivalently, a constant $\lambda \in \mathbb{R}$ is an evalue of A iff $(x-\lambda) \mid \mathrm{char}_A(x)$.
- For each evalue λ of A, the algebraic multiplicity of λ is the multiplicity of the factor $(x-\lambda)$ in $\mathrm{char}_A(x)$.
  The corresponding eigenvectors v w/ $Av = \lambda v$ are exactly the nonzero elts in $\mathrm{Null}(A - \lambda I_n)$.

Today: · finding eigenvectors / studying $E_\lambda := \{ v \mid Av = \lambda v \} = \mathrm{Null}(A - \lambda I_n)$.

1. Eigenspaces

Def: (eigenspace, geometric multiplicity) For each $\mu \in \mathbb{R}$, we define

$E_\mu = \underbrace{\{v \in \mathbb{R}^n \mid Av = \mu v\}}_{0}$. Thus, $E_\mu \neq \{0\} \iff \mu$ is an e-value of A.

When μ is an e-value (ie. $E_\mu \neq 0$), we call E_μ the eigenspace of the e-value μ

and we call $\dim E_\mu$ the geometric multiplicity of μ.

Note: · Recall from last time that $E_\lambda = \mathrm{Null}(A - \lambda I_n)$.

· In particular, we know how to find a basis of E_λ, and if λ is an evalue then its geom. mult is $\dim E_\lambda = \dim \mathrm{Null}(A - \lambda I_n)$

We'll be interested in finding a basis of E_λ. $= \#$ nonpivot cols in $(A - \lambda I_n)$.

Eg. For each matrix below, find it eigenvalues, find the alg. & geom. multiplicities of each evalue, and find a basis for each espace.

(1). $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$

Soln: $\text{char}_A(x) = \begin{vmatrix} 2-x & 3 \\ 3 & -6-x \end{vmatrix} = (2-x)(-6-x) - 9 = -12 + 6x - 2x + x^2 - 9$

$= x^2 + 4x - 21 = (x+7)(x-3)$

so the roots of $\text{char}_A(x)$, hence the e-values, are $\lambda_1 = -7$ and $\lambda_2 = 3$; they both have alg. mult. 1.

$\lambda_1 = -7$. $E_{\lambda_1} = \text{Null}(A - \lambda_1 I_2) = \text{Null}\left(\begin{bmatrix} 2+7 & 3 \\ 3 & -6+7 \end{bmatrix}\right) = \text{Null}\left(\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}\right)$

$\left[\begin{array}{cc|c} 9 & 3 & 0 \\ 3 & 1 & 0 \end{array}\right] \to \begin{bmatrix} 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 \end{bmatrix} \to E_{\lambda_1} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x + \frac{1}{3}y = 0 \right\} = \left\{ \begin{bmatrix} -\frac{1}{3}y \\ y \end{bmatrix} = y \cdot \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix} \,\middle|\, y \in \mathbb{R} \right\}.$

Thus, E_{λ_1} has a basis $\left\{\begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}\right\}$ and dim. 1.

So λ_1 has geom. mult. 1. (alg. mult = geom. mult $=1$ for λ_1)

$\underline{\lambda_2 = 3.}$ $E_{\lambda_2} = Null(A - \lambda_2 I_2) = Null\left(\begin{bmatrix} 2-3 & 3 \\ 3 & -6-3 \end{bmatrix}\right) = Null\left(\begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix}\right)$

$$\begin{bmatrix} -1 & 3 & 0 \\ 3 & -9 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to E_{\lambda_2} = \left\{\begin{bmatrix} x \\ y \end{bmatrix} : x - 3y = 0\right\}$$

$$= \left\{\begin{bmatrix} 3y \\ y \end{bmatrix} = y\begin{bmatrix} 3 \\ 1 \end{bmatrix} \;\middle|\; y \in \mathbb{R}\right\}.$$

Thus, E_{λ_2} has a basis $\left\{\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right\}$ and dim 1.

So λ_2 has geom. mult 1. (again, alg mult = geom. mult = 1 for λ_2)

$\square$

(2). $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$.

Soln: Recall from the last lecture that $\text{char}_A(x) = (1-x)(x+2)^2$, so the evalues are $\lambda_1 = 1$ and $\lambda_2 = -2$, which have alg. mult. 1 and 2, resp.

$\lambda_1 = 1$ $\quad E_{\lambda_1} = \text{Null}(A - \lambda_1 I_3) = \text{Null}\left(\begin{bmatrix} 1-1 & 3 & 3 \\ -3 & -5-1 & -3 \\ 3 & 3 & 1-1 \end{bmatrix}\right) = \text{Null}\left(\begin{bmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{bmatrix}\right)$

$$\begin{bmatrix} 0 & 3 & 3 & 0 \\ -3 & -6 & -3 & 0 \\ 3 & 3 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\to E_{\lambda_1} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : \begin{matrix} x - z = 0 \\ y + z = 0 \\ z \in \mathbb{R} \end{matrix} \right\} = \left\{ \begin{bmatrix} z \\ -z \\ z \end{bmatrix} = z \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} : z \in \mathbb{R} \right\}$$

So E_{λ_1} has a basis $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$ and dim 1; λ_1 has geom. mult. 1.

$\underline{\lambda_2 = -2}$. $E_{\lambda_2} = \text{Null}\left(\begin{bmatrix} 1+2 & 3 & 3 \\ -3 & -5+2 & -3 \\ 3 & 3 & 1+2 \end{bmatrix}\right) = \text{Null}\left(\begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix}\right)$

$$\left[\begin{array}{ccc|c} 3 & 3 & 3 & 0 \\ -3 & -3 & -3 & 0 \\ 3 & 3 & 3 & 0 \end{array}\right] \to \begin{array}{c} \left[\begin{array}{cccc} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cccc} \mid & \mid & \mid & \\ p & f. & f. & \end{array} \end{array} \to E_{\lambda_2} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x+y+z=0 \right\}$$

$$= \left\{ \begin{bmatrix} -y-z \\ y \\ z \end{bmatrix} \,\middle|\, y, z \in \mathbb{R} \right\} = \left\{ y\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} : y, z \in \mathbb{R} \right\}.$$

So E_{λ_2} has a basis $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ and dim 2, meaning the geom. mult of λ_2 is 2. $\square$

<u>Note:</u> The alg. mult. and geom mult. coincide for both λ_1 and λ_2.

13). $A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$. (E.g. 5.3.4)

Soln: E.X: show that $\text{char}_A(x) = (1-x)(x+2)^2$, so that A has evalues $\lambda_1 = 1$, $\lambda_2 = -2$ w/ alg. mult. 1 and 2, respectively.

$\lambda_1 = 1$: $E_{\lambda_1} = \text{Null}\left(\begin{bmatrix} 2-1 & 4 & 3 \\ -4 & -6-1 & -3 \\ 3 & 3 & 1-1 \end{bmatrix}\right) = \text{Null}\left(\begin{bmatrix} 1 & 4 & 3 \\ -4 & -7 & -3 \\ 3 & 3 & 0 \end{bmatrix}\right)$

$\rightarrow \ldots$ E_{λ_1} has basis $\left\{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right\}$, so E_{λ_1} has dim 1 and $\text{geom.mult}(\lambda_1) = 1$.

$$\text{alg.mult}(\lambda_1) \\ \shortparallel$$

$\lambda_2 = -2$: $E_{\lambda_2} = \text{Null}\left(\begin{bmatrix} 4 & 4 & 3 \\ -4 & -4 & -3 \\ 3 & 3 & 3 \end{bmatrix}\right)$; $\begin{bmatrix} 4 & 4 & 3 & 0 \\ -4 & -4 & -3 & 0 \\ 3 & 3 & 3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 4 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 1 & 1 & 1 & 0 \\ 4 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{array}{c} \quad f \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \Rightarrow E_{\lambda_2} = \left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} : \begin{array}{l} x+y=0 \\ -z=0 \\ y \in \mathbb{R} \end{array}\right\} = \left\{\begin{bmatrix} -y \\ y \\ 0 \end{bmatrix} = y\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \,\middle|\, y \in \mathbb{R}\right\}$

Thus, E_{λ_2} has a basis $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$ and dim 1, so that the geom. mult of λ_2 is 1. $\square$

illustrates

$$2 = \text{alg. mult}(\lambda_2) > \text{geom. mult}(\lambda_2) = 1 > 0.$$

Thm: Let A be an $n \times n$ matrix. Then for every evalue λ of A, we have

(i). geom. mult $(\lambda) \geq 1$ (since $E_\lambda \neq \{0\}$ so geom. mult $(\lambda) > 0$)

(ii). alg. mult $(\lambda) \geq$ geom. mult. (λ), but the equality doesn't have to hold.

Note: (i) and (ii) imply that if alg. mult $(\lambda) = 1$ then geom. mult (λ) must be 1; in particular, any non zero vector we can find in E_λ would form a basis of E_λ by itself.

Furthermore, if the alg mult. and geom mult. of λ coincide for every evalue λ of A, then the union of the bases of the espaces forms a basis of $\mathbb{R}^n$.

it has nice properties.

e.g. Our Eg. (2), $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a "nice" basis of $\mathbb{R}^3$.

E_{λ_1} E_{λ_2}

What does "nice" mean here?

— next time!