Math 2130. Lecture 33. 04.07.2021.

Last time: • defined eigenvectors and eigenvalues of square matrices.

• motivation for studying eigenvectors

Def: Let A be a square matrix. An eigenvector of A is a nonzero vector v to s.t. $Av = \lambda v$ for some scalar $\lambda \in \mathbb{R}$. Such a scalar is called an eigenvalue of A.

Note: An e-vec. of e-value 0 is just a nonzero elt s.t. $Av = 0 \cdot v = 0$, i.e., just a nonzero elt in $\mathrm{Nul}\,A$.

E.g.: If $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, $v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $w = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, then

$Av = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \neq \lambda \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ for any $\lambda \in \mathbb{R}$, $Aw = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \cdot w$

So v is not an e-vect. of A, but w is and has eigenvalue 0.

Today: · the characteristic polynomial · finding eigenvalues.

An important observation: Let A be an $n \times n$ matrix and let $T: \mathbb{R}^n \to \mathbb{R}^n$ be the associated map given by $T(v) = A \cdot v$.

For any constant $\lambda \in \mathbb{R}$ and vector $v \in \mathbb{R}^n$, we have

$$Av = \lambda v \iff Av = (\lambda \cdot I_n) \cdot v \iff (A - \lambda I_n) \cdot v = 0 \iff v \in \text{Null}(A - \lambda I_n)$$

In particular, λ is an eigenvalue of A $\iff$ there is a nonzero $v \in \text{Null}(\underbrace{A - \lambda I_n}_{n \times n \text{ matrix}})$

$\iff$ $A - \lambda I_n$ has a nontrivial null space

$\iff$ $A - \lambda I_n$ is not inv $\iff$ $\det(A - \lambda I_n) = 0$.

Conclusion: $\forall x \in \mathbb{R}$, x is an eigenvalue of A $\iff$ $\det(A - x I_n) = 0$.

Upshot: (i) $\forall x \in \mathbb{R}$, x is an eigenvalue of $A \iff \det(A - xI_n) = 0$.

(ii) If $\lambda \in \mathbb{R}$ is an eigenvalue of A, then v is a corr. e-vec of A

$\Updownarrow$

$v \in Null(A - \lambda I_n)$.

The observations will allow us to

(1). find all e-values of A $\to$ today.

(2). for each eigenvalue of A, find all corresponding eigenvectors.

What's $\det(A - xI)$ like? e.g., $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \to \det \begin{bmatrix} 1-x & 2 \\ 3 & 4-x \end{bmatrix} = (1-x)(4-x) - 2\cdot 3$

Answer: It's a polynomial in x.

Def. We call the polynomial $char_A(x) = \det(A - xI)$ the <u>characteristic polynomial</u> of A

Thm. (By (i)) A scalar λ is an e-value of $A \iff \lambda$ is a root of $char_A(x)$.

E.g. Find all e-values of $A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$.

Soln: We compute $\text{char}_A(x) = \det \begin{bmatrix} 5-x & 3 \\ 3 & 5-x \end{bmatrix} = (5-x)^2 - 3 \cdot 3$

$$= 25 - 10x + x^2 - 9$$

$$= x^2 - 10x + 16$$

The e-values of A are 2 and 8.

To find the e-values of A, we can

- either use the quadratic formula to find the roots of $\text{char}_A(x)$:

$ax^2+bx+c=0 \Rightarrow \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$ $\quad \dfrac{10 \pm \sqrt{100-64}}{2} = \dfrac{10 \pm \sqrt{36}}{2} = \dfrac{10 \pm 6}{2} = 8$ or 2.

- or factor $\text{char}_A(x)$ to find the roots.

$\text{char}_A(x) = x^2 - 10x + 16 = (x-8)(x-2)$. $\Rightarrow$ the roots are 2 and 8.

Eg. Find all e-values of $B = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$.

Soln:

$$\text{char}_B(x) = \det \begin{bmatrix} 1-x & 3 & 3 \\ -3 & -5-x & -3 \\ 3 & 3 & 1-x \end{bmatrix} = (1-x)\begin{vmatrix} -5-x & -3 \\ 3 & 1-x \end{vmatrix} - 3\cdot\begin{vmatrix} -3 & -3 \\ 3 & 1-x \end{vmatrix}$$

$$+ 3\cdot\begin{vmatrix} -3 & -5-x \\ 3 & 3 \end{vmatrix}$$

$$= (1-x)\left[(-5-x)(1-x) - 3\cdot(-3)\right] - 3\left[-3(1-x) - 3\cdot(-3)\right]$$

$$+ 3\cdot\left[-3\cdot 3 - 3\cdot(-5-x)\right]$$

$$= (1-x)\left[-5-x+5x+x^2+9\right] - 3\cdot\left[-3+3x+9\right] + 3\cdot\left[-9+15+3x\right]$$

$$= (1-x)\underbrace{(x^2+4x+4)}_{(x+2)^2} \cancel{- 3(3x+6)} \cancel{+3(3x+6)} = (1-x)(x+2)^2$$

Since $\text{char}_B(x) = (1-x)(x+2)^2$, it has roots 1 and -2.

$-(x-1)(x-(-2))^2$

So the eigenvalues of B are 1 and -2.

Def. (algebraic multiplicity) Note that λ is an evalue of $A \iff \lambda$ is a root of $\text{char}_A(x) \overset{\text{algebra}}{\iff} (x-\lambda)$ is a linear factor of $\text{char}_A(x)$.

The multiplicity with which the term $(x-\lambda)$ appears in $\text{char}_A(x)$ for a e-value λ is called the algebraic multiplicity of λ.

E.g. In our first example, the e-values 2 and 8 of A each has alg. mult. 1.

In the second example, the evalues 1 and -2 have alg. mult. 1 and 2, resp.

Now we know how to find eigenvalues of a square matrix A.

If we know λ is an eigenvalue, how do we find its e-vectors?

Recall: given an e-value λ of A, v is a corr. e-vect $\iff 0 \neq v \in \underline{\mathrm{Null}(A-\lambda I)}$.

we know how to find it.

Eg. Take our $A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$ and its e-value 2.

v is an e-vec of e-value $\lambda = 2$ $\iff 0 \neq v \in \mathrm{Null}(A - 2I) = \mathrm{Null}\left(\begin{bmatrix} 5-2 & 3 \\ 3 & 5-2 \end{bmatrix}\right) = \mathrm{Null}\left(\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}\right)$.

$$\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} 3 & 3 & 0 \\ 3 & 3 & 0 \end{bmatrix} \to \begin{array}{cc} \begin{array}{ccc} x & y & \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$x + y = 0 \to \begin{bmatrix} -y \\ y \end{bmatrix} = y\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$= \underbrace{\left\{ y\begin{bmatrix} 1 \\ -1 \end{bmatrix} : y \in \mathbb{R} \right\}}_{= S}$$

the e-vectors w/ e-value 2 are just the nonzero elts in S.

So: To find eigenvectors of a given e-value is really to find a null space.

$\to$ next time: more of these computations.