Math 2130. Lecture 32.

04. 05. 2021.

Last time: eigenvalues and eigenvectors

Today: midterm review.

Review Problems:

7. (a). $d: P_3 \to P_3$ "diff": $d(\underbrace{a_0 + a_1 t + a_2 t^2 + a_3 t^3}_{})$

$$= \underbrace{0 + a_1 + 2a_2 t + 3a_3 t^2}_{}.$$

Why is d linear: check $d(f+g) = d(f) + d(g)$, i.e., $\underbrace{(f+g)' = f' + g'}_{(i)}$ $\forall f, g \in P_3$

$d(cf) = c \cdot d(f)$, i.e., $\underbrace{(cf)' = c \cdot f'}_{(ii)}$ $c \in \mathbb{R}$.

(i) & (ii) hold by calculus, so d is linear.

(b).

$$\ker(d) = \{ f : d(f) = 0 \} = \{ a_0 + a_1 t + a_2 t^2 + a_3 t^3 \mid \underbrace{a_1 + 2a_2 t + 3a_3 t^2}_{= 0} \}$$

$$= \{ a_0 + a_1 t + a_2 t^2 + a_3 t^3 \mid a_1 = 0,\ a_2 = 0,\ a_3 = 0 \}$$

$$= \{ a_0 \mid a_0 \in \mathbb{R} \} \equiv \{ \text{the constant polynomials} \}.$$

So ker d consists of exactly all the constant poly. in P_3.

The set $\{1\}$ is a basis of ker(d) since it spans $\widehat{\ }$ker(d) (any constant poly c is $c \cdot 1$) and is lin ind (...).

So the dim. of ker(d) is 1.

(c). similar.

(d). Recall that P_3 has a standard basis $\beta = \{1, t, t^2, t^3\}$.

$[\ ]_\beta : P_3 \to \mathbb{R}^4$.

e.g. $2+t \mapsto \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

So $B = \{v_1, v_2, v_3, v_4\}$ is a basis in P_3

Iff $\beta' = \{[v_1]_\beta, [v_2]_\beta, [v_3]_\beta, [v_4]_\beta\}$ is a basis of $\mathbb{R}^4$.

check whether this is true using any method we've covered.

(e). Note that $[g]_B = [g]_{\{v_1, v_2, v_3, v_4\}} = \left[[g]_\beta \right]_{\{[v_1]_\beta, [v_2]_\beta, [v_3]_\beta, [v_4]_\beta\}}$.

$\begin{bmatrix} -10 \\ 1 \\ 2 \\ 1 \end{bmatrix}$

8. (a). Use your favorite criterion for basis status.

e.g. $[b_1 | b_2] = \begin{bmatrix} 7 & -3 \\ 5 & 1 \end{bmatrix}$, $\longrightarrow \det \begin{bmatrix} 7 & -3 \\ 5 & 1 \end{bmatrix} = 7 \times 1 - (-3) \times 5 = 22 \neq 0.$

so $\mathcal{B} = \{b_1, b_2\}$ is a basis.

(b). $\underset{C \leftarrow B}{P}$: use our algorithm $[\, c_1 | c_2 \,|\, b_1 | b_2 \,] \xrightarrow{\text{red.}} \left[\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \,\middle|\, M \right]$

$\Downarrow$

$M = \underset{C \leftarrow B}{P}$.

(c). $\underset{B \leftarrow C}{P}$ using (b) : $\underset{B \leftarrow C}{P} = \left(\underset{C \leftarrow B}{P} \right)^{-1}$.

$\downarrow$ if not required to use (b), can apply the same alg. on $[\, b_1 | b_2 \,|\, c_1 | c_2 \,]$.

(d). $C, [v]_C \longrightarrow v$, easy : $v = 1\cdot c_1 + 3\cdot c_2 = 1\cdot\begin{bmatrix}1\\-5\end{bmatrix} + 3\begin{bmatrix}-2\\2\end{bmatrix} = \begin{bmatrix}-5\\1\end{bmatrix}$.

$v, B \longrightarrow [v]_B$, hard, but there are two possible methods here.

(1). Solve the system w/ aug matrix $\left[\begin{array}{c|c|c} b_1 & b_2 & v \end{array}\right]$

$$x\begin{bmatrix}7\\5\end{bmatrix} + y\begin{bmatrix}-3\\-1\end{bmatrix} = \begin{bmatrix}-5\\1\end{bmatrix}.$$

(2). $[v]_B = \underset{B\leftarrow C}{P}\,[v]_C = \begin{bmatrix}-2 & 1\\-5 & 3\end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}1\\4\end{bmatrix}$.

(c).

6. (a). state the necessary conditions.

(b). verify the conditions for $\operatorname{Im} T$.

(i) "has zero": Recall that since T is linear, we must have

$$T(0_V) = \underline{\underline{0_W}}$$

so $0_W \in \operatorname{Im} T$.

(ii). "closure under addition": suppose $w_1, w_2 \in \operatorname{Im} T$. Then

by linearity

$w_1 = T(v_1)$, $w_2 = T(v_2)$ for some $v_1, v_2 \in V$, so $w_1 + w_2 = T(v_1) + T(v_2) = T(v_1 + v_2)$

so $w_1 + w_2 \in \operatorname{Im} T$. (iii). "closure under scaling": similar...