Math 2130. Lecture 31.

Midterm II : next Monday

Soln to the review: to be posted on Sunday

04.02.2021.

Last time:

· matrices of linear maps

$$\begin{tikzcd}[column sep=huge, row sep=large]
W \ni T(v) \arrow[d, "{[\ ]_C,\ C \text{ a basis of } W}"'] & V \ni v \arrow[l, "T"'] \arrow[d, "{[\ ]_B,\ B \text{ a basis of } V}"] \\
{[T(v)]_C} & {[v]_B} \arrow[l, "{[T]_B^C}"']
\end{tikzcd}$$

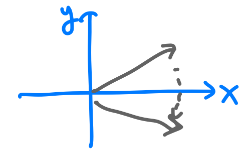
$[T]_B^C$: the matrix of T w.r.t B and C.

Today:

· first look at eigenvectors / eigenvalues.

Motivation: Consider the following setting: we discuss linear maps $T: V \to V$ from a v.s. V to itself. We are interested in the matrix $[T]^{B}_{B}$ where B are various bases of V. (e.g. $V = \mathbb{R}^3$, $\to$ $B_1 = \{e_1, e_2, e_3\}$. $B_2 = \left\{ \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\2\end{bmatrix} \right\}$)

E.g.1. $V = \mathbb{R}^2$, T = refl. w.r.t the x-axis



$\begin{bmatrix}x\\y\end{bmatrix} \mapsto \begin{bmatrix}x\\-y\end{bmatrix}$

$$B_1 = \left\{ \underset{v_1}{\begin{bmatrix}1\\0\end{bmatrix}}, \underset{v_2}{\begin{bmatrix}0\\1\end{bmatrix}} \right\} \qquad B_2 = \left\{ \underset{w_1}{\begin{bmatrix}1\\0\end{bmatrix}}, \underset{w_2}{\begin{bmatrix}2\\1\end{bmatrix}} \right\}$$

because $\begin{bmatrix}2\\-1\end{bmatrix} = 4w_1 - w_2$

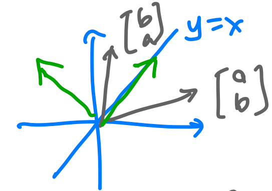
$$[T]^{B_1}_{B_1} = \Big[[T(v_1)]_{B_1} \,\Big|\, [T(v_2)]_{B_1} \Big] = \left[\left[\begin{bmatrix}1\\0\end{bmatrix}\right]_{B_1}, \left[\begin{bmatrix}0\\-1\end{bmatrix}\right]_{B_1} \right] = \underline{\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}}$$

diagonal

$$[T]^{B_2}_{B_2} = \Big[[T(w_1)]_{B_2} \,\Big|\, [T(w_2)]_{B_2} \Big] = \left[\left[\begin{bmatrix}1\\0\end{bmatrix}\right]_{B_2} \,\middle|\, \left[\begin{bmatrix}2\\-1\end{bmatrix}\right]_{B_2} \right] = \begin{bmatrix}1 & 4\\0 & -1\end{bmatrix}$$

"lodes nizer"

E.g. $V=\mathbb{R}^2$, $T=$ refl. w.r.t the line $y=x$.



$$\begin{bmatrix}x\\y\end{bmatrix}\mapsto\begin{bmatrix}y\\x\end{bmatrix},$$

$B_1=\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}$ $\rightarrow$

$$[T]^{B_1}_{B_1}=\left[\left[T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)\right]_{B_1}\middle|\left[T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)\right]_{B_1}\right]$$

$$=\left[\left[\begin{bmatrix}0\\1\end{bmatrix}\right]_{B_1}\middle|\left[\begin{bmatrix}1\\0\end{bmatrix}\right]_{B_1}\right]=\begin{bmatrix}0&1\\1&0\end{bmatrix}.$$ $\rightarrow$ not diag.

$B_2=\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}-1\\1\end{bmatrix}\right\}$ $\rightarrow$

$$[T]^{B_2}_{B_2}=\left[\left[T\left(\begin{bmatrix}1\\1\end{bmatrix}\right)\right]_{B_2}\middle|\left[T\left(\begin{bmatrix}-1\\1\end{bmatrix}\right)\right]_{B_2}\right]=\left[\underline{\left[\begin{bmatrix}1\\1\end{bmatrix}\right]_{B_2}},\left[\begin{bmatrix}1\\-1\end{bmatrix}\right]_{B_2}\right]$$

$$=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$$ $\rightarrow$ diagonal "nice".

Point: if we select a basis B of V carefully, we will sometimes be able to ensure that $[T]^B_B$ is diagonal.

Q: What makes this carefully selected basis special? A: It's special because its elts are sent to their multiples of themselves.

Note: Given a linear map $T: \mathbb{R}^n \to \mathbb{R}^m$. We introduced the standard matrix of T as the matrix $A_T = \left[\, T(e_1) \;\vdots\; \cdots \;\vdots\; T(e_n) \,\right]$

Claim: $A_T = [T]^{C}_{B}$ where B is the standard matrix of $\mathbb{R}^n$ and C is the standard matrix of $\mathbb{R}^m$.

Ex: Make sense of and prove the claim. (E.g. think about $T: \mathbb{R}^3 \to \mathbb{R}^2$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} x+3y \\ z-y \end{bmatrix}.$$)

Eigenvectors and eigenvalues

Let A be an $n \times n$ matrix.

Def: (eigenvector and eigenvalue of a matrix) An eigenvector of A is a vector $v \in \mathbb{R}^n$ s.t. (1) $v \neq 0$ and (2) $Av = \lambda v$ for some constant $\lambda \in \mathbb{R}$.

In this case, the constant λ is called an eigenvalue of A and we say that v is an eigenvector of A corresponding to λ.

compare this w/ Page 3.

E.g. Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are eigenvectors of A corresponding to eigenvalues 1 and -1, respectively:

$$Av_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = v_1 \qquad Av_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -v_2.$$

E.g. Let $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$. Are the vectors $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$

$w = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

eigenvectors of A ?

Soln:

$$Av_1 = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix} = 5 v_1 .$$

$$Av_2 = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 1 \cdot v_2 .$$

$$Av_3 = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 1 \cdot v_3 .$$

$\Downarrow$

$$Aw = \begin{bmatrix} 9 \\ 10 \\ 11 \end{bmatrix} \neq \lambda \cdot w$$

for any λ.

So w is not an e-vec of A.

So v_1, v_2, v_3 are e-vec. of A with e-values $5, 1, 1$, respectively.

Next time: Given A, we will be able to find all the e-values of A

and then all the e-vectors corr. to each e-value.

We'll learn how after the midterm.