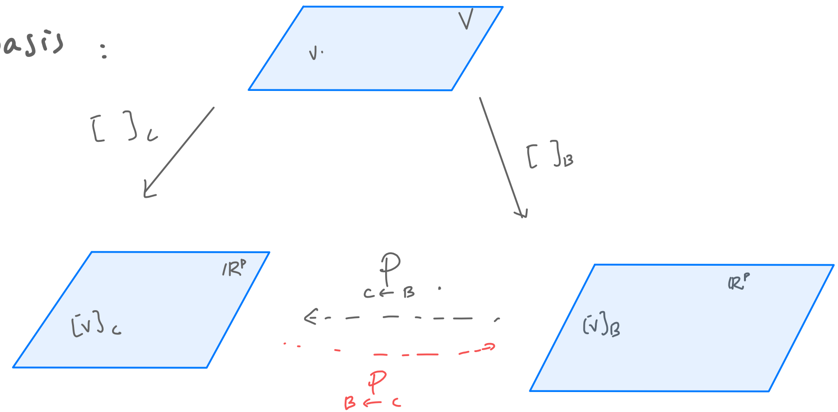
# Math 2130. Lecture 30.

03. 31. 2021.

**Last time:** · change of basis:




· finding the change of basis matrix $\underset{C \leftarrow B}{P} = \text{``}\left[ [b_1]_C \,\middle|\, \cdots \,\middle|\, [b_p]_C \right]\text{''}$.

observation: in finding $[b_i]_C$ for all $i$, we use the same row op.s to reduce $C$ to $I_p$.

**Today:** 1. algorithm for finding $\underset{C \leftarrow B}{P}$ in $V = \mathbb{R}^p$ 2. matrix for linear maps

## 1. Algorithm for finding

Let $B = \{b_1, b_2, \cdots, b_p\}$ and $C = \{c_1, c_2, \cdots, c_p\}$ be bases of $\mathbb{R}^p$.

By the observation, we can find the change-of-basis matrix $\underset{C \leftarrow B}{P}$ as follows:

<u>Thm:</u> To find $\underset{C \leftarrow B}{P}$, it suffices to row reduce the matrix

$$\text{“}[\, C \mid B \,]\text{”} = \left[\, \underbrace{c_1 \mid c_2 \mid \cdots \mid c_p}_{\text{the “left half”}} \;\middle|\; b_1 \mid b_2 \mid \cdots \mid b_p \,\right]$$

until the left half becomes $I_p$. At this point the right half will be $\underset{C \leftarrow B}{P}$.

E.g. $V = \mathbb{R}^2$, $\mathcal{B} = \left\{ \underset{b_1}{\begin{bmatrix} -2 \\ 1 \end{bmatrix}}, \underset{b_2}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \right\}$ and $\mathcal{C} = \left\{ \underset{c_1}{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}, \underset{c_2}{\begin{bmatrix} 3 \\ 1 \end{bmatrix}} \right\}$. Find $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$

↘ the last ...

Soln: $$[\,C \mid B\,] = \left[\begin{array}{cc|cc} 2 & 3 & -2 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|cc} 1 & \frac{3}{2} & -1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cc|cc} 1 & 0 & -\frac{5}{2} & -\frac{3}{2} \\ 0 & 1 & 1 & 1 \end{array}\right]$$

$I_2$. done!

It follows that $$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \begin{bmatrix} -\frac{5}{2} & -\frac{3}{2} \\ 1 & 1 \end{bmatrix}.$$

Ex. Use the algorithm to find $\underset{\mathcal{B} \leftarrow \mathcal{C}}{P}$, then check that $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \left(\underset{\mathcal{B} \leftarrow \mathcal{C}}{P}\right)^{-1}$.

Eg. $V = \mathbb{R}^2$, $\quad C = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$, $\quad B = \left\{ \begin{bmatrix} 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \end{bmatrix} \right\}$. Find $\underset{C \leftarrow B}{P}$.

Soln:

$$[\, C \mid B \,] := \begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -2 & -4 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & \underline{2} & \underline{3} \end{bmatrix}, \text{ so } \underset{C \leftarrow B}{P} = \begin{bmatrix} -1 & -2 \\ 2 & 3 \end{bmatrix}.$$

$\qquad\qquad\qquad [b_1]_C \quad [b_2]_C$

Verification:

(a). Is $[b_1]_C = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$? $\quad -1 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix} = b_1$ ✓

(b) Is $[b_2]_C = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$? $\quad -2 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3 \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix} = b_2$ ✓

Eg. $V = \mathbb{R}^3$, $\underbrace{C = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}}_{\text{standard basis}}$, $B = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 10 \end{bmatrix} \right\}$.

Note: $[C \mid B] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 4 & 7 \\ 0 & 1 & 0 & 2 & 5 & 8 \\ 0 & 0 & 1 & 3 & 6 & 10 \end{array}\right]$

already $I_3$ since $C$ is standard

$$\rightarrow \underset{C \leftarrow B}{P} = \underset{\substack{\text{stand.} \leftarrow B \\ \text{basis}}}{P} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 10 \end{bmatrix} = [b_1 \mid b_2 \mid b_3]$$

Point: For any basis $B = \{b_1, \dots, b_p\}$ of $\mathbb{R}^p$, if $C$ is the standard matrix the $\underset{C \leftarrow B}{P} = [b_1 \mid \cdots \mid b_p]$.

More examples in HW.

## 2. Matrix of a linear map. (with respect to a chosen basis for the domain and a chosen basis for the codomain.)

**Def:** Let $V, W$ be (finite-dimension) vec. spaces. Let $T: V \to W$ be a linear map. Let $B = \{b_1, b_2, \dots, b_n\}$ be a basis of $V$ and $C = \{c_1, \dots, c_m\}$ be a basis of $W$. The matrix of $T$ relative to $B$ and $C$ is the $\underset{m \times n}{\text{matrix}}$

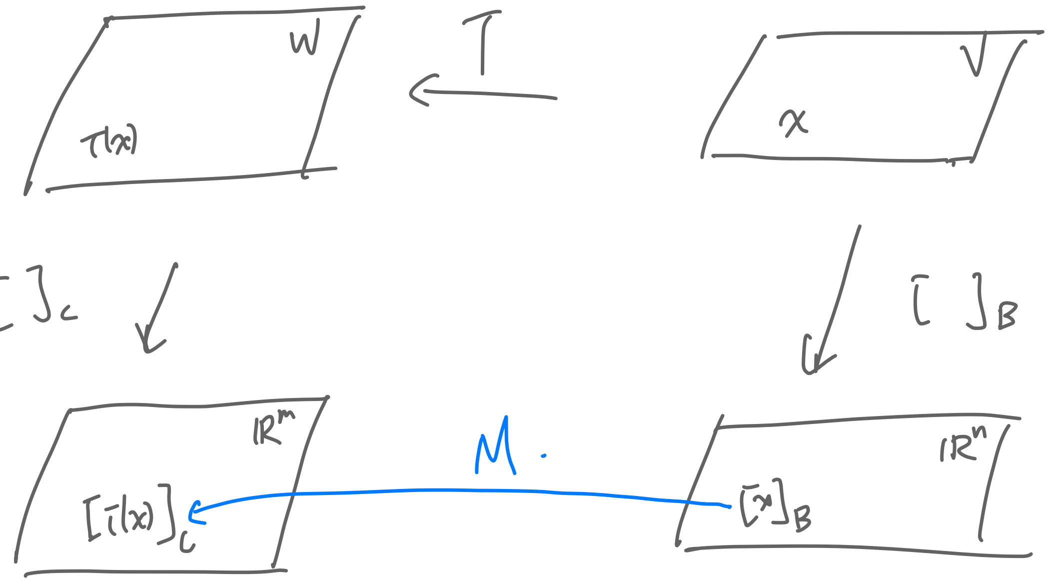
$$M = \begin{bmatrix} \Big| & & \Big| \\ [T(b_1)]_C & \cdots & [T(b_n)]_C \\ \Big| & & \Big| \end{bmatrix}.$$

(the $j$th column should record the $C$-decomp. of $T(b_j)$)

**Thm.** In the above setting we have

$$[T(x)]_C = M \cdot [x]_B \qquad \underline{\underline{\forall}}\, x \in V.$$

**Picture:**




**Notation:** We'll often denote $M$ by $[T]_B^C$.

**Note:** Everything should be done relative to the chosen bases.

E.g. Suppose that $V$ has a basis $B=\{b_1, b_2\}$ and $W$ has a basis $C=\{c_1, c_2, c_3\}$. Suppose $T: V \to W$ is the linear map such that

$$T(b_1) \overset{①}{=} 3c_1 - 2c_2 + 5c_3, \quad T(b_2) \overset{②}{=} 4c_1 + 7c_2 - c_3$$

Then $$[T]_B^C = \left[ [T(b_1)]_C \;\middle|\; [T(b_2)]_C \right] = \begin{bmatrix} 3 & 4 \\ -2 & 7 \\ 5 & -1 \end{bmatrix}.$$

Verify thm for $x = 2b_1 - 3b_2$: we have

(i) $[x]_B = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$, $\Rightarrow$ $[T]_B^C [x]_B = \begin{bmatrix} 3 & 4 \\ -2 & 7 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -6 \\ -25 \\ 13 \end{bmatrix} \overset{✓}{=} [T(x)]_C$

(ii) $T(x) = T(2b_1 - 3b_2) = 2T(b_1) - 3T(b_2) \overset{①,②}{=} 2(3c_1 - 2c_2 + 5c_3) - 3(4c_1 + 7c_2 - c_3)$

$= (2\cdot 3 - 3\cdot 4)c_1 + (2\cdot(-2) - 3\cdot 7)c_2 + (2\cdot 5 - 3\cdot(-1))c_3 = -6c_1 - 25c_2 + 13c_3$

so $[T(x)]_C = \begin{bmatrix} -6 \\ -25 \\ 13 \end{bmatrix}$

## Connection to change-of-basis matrices :

When $V = W$ and $T = id_V$. (i.e. $T(x) = x \ \forall x \in V$), the picture from early becomes

$$\begin{tikzcd} V \ni v \arrow[d, "{[\ ]_C}"'] & V \ni v \arrow[l, "T=\mathrm{Id}_V"'] \arrow[d, "{[\ ]_B}"] \\ \mathbb{R}^n \ni [v]_C & \mathbb{R}^n \ni [v]_B \arrow[l, "{[T]^C_B}"] \end{tikzcd}$$

or

$$\begin{tikzcd} & V \ni v \arrow[dl, "{[\ ]_C}"'] \arrow[dr, "{[\ ]_B}"] & \\ \mathbb{R}^n \ni [v]_C & & \mathbb{R}^n \ni [v]_B \arrow[ll, "{[T]^C_B}"] \end{tikzcd}$$

therefore, by the property of $\underset{C \leftarrow B}{P}$ and its uniqueness.

$$[id_V]^C_B \overset{\checkmark}{=} \underset{C \leftarrow B}{P}.$$

$\Big($ indeed $\underset{C \leftarrow B}{P} = \big[ [b_1]_C \big| \cdots \big| [b_n]_C \big]$

$= \big[ [Id(b_1)]_C \big| \cdots \big| [Id(b_n)]_C \big] = [Id]^C_B \Big).$

**E.g.** Take $V = P_3$, $W = P_2$, $T: V \to W$ the unique linear map

s.t. $T(t^n) = n t^{n-1}$ $\forall n \in \{0, 1, 2, 3\}$.

Take $B = \{1, t, t^2, t^3\} \subseteq P_3$, $C = \{1, t, t^2\}$. Then

$$[T]_B^C = \left[ [T(1)]_C \,\middle|\, [T(t)]_C \,\middle|\, [T(t^2)]_C \,\middle|\, [T(t^3)]_C \right] = \left[ [0]_C \,\middle|\, [1]_C \,\middle|\, [2t]_C \,\middle|\, [3t^2]_C \right]$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Let $C' = \{2, 2t+2, t^2\}$. Then

diff. since $C \neq C'$.

$$[T]_B^{C'} = \left[ [T(1)]_{C'} \,\middle|\, \cdots \,\middle|\, [T(t^3)]_{C'} \right] = \left[ [0]_{C'} \,\middle|\, [1]_{C'} \,\middle|\, [2t]_{C'} \,\middle|\, [3t^2]_{C'} \right] = \begin{bmatrix} 0 & \frac{1}{2} & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$