Math 2130. Lecture 3.

01/22/2021

Last time: · Def. of elementary row operations:

E1. interchange E2. scaling E3. replacement

& (reduced) echelon forms: EF: (1) zero rows below nonzero rows.

(2) "staircase condition"; REF: (1) + (2) + (a): row leading entries are all 1

+ (b): row leading entries are the only nonzero entries in their columns.

· Thm: Using elt. row operations, we can transform every matrix A to an echelon form and a unique reduced echelon form.

Today.

1. Gaussian elimination: algorithm to transform a matrix to (reduced) echelon form using elt. row ops.

The algorithm:

Step 1. Find the leftmost nonzero column. Interchanging rows if necessary, make sure the top entry in that column is nonzero. (e.g. $\begin{bmatrix} 0 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix} \xrightarrow{E1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 3 \end{bmatrix}$)

Step 2. Use E3 to create zeros below the top nonzero entry from step 1.

running example.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \xrightarrow[-5R_1]{R2 \leftarrow R2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & (6-2\cdot5) & (7-3\cdot5) & (8-4\cdot5) \\ 9 & 10 & 11 & 12 \end{bmatrix} \xrightarrow{R3 \leftarrow R3-9R1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -24 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

Step 3. Repeat steps (1) and (2) on the submatrix to the lower right of the nonzero entry from Step (1).

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \xrightarrow{(1)\ \&\ (2)} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -24 \end{bmatrix}$$

$$\rightarrow \begin{array}{c} \\ \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -24 \end{bmatrix} \end{array} \begin{array}{l} \\ R_x \\ R_y \end{array} \xrightarrow{R_y \leftarrow R_y - 2R_x} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \text{all zeros, halt!}$$

Repeat this process until the resulting submatrix is empty or contains only zeros.

This results in an echelon form.

Step 4. (for reduced echelon form)

To get the resulting echelon form step (3) into a reduced echelon form, use E2 to make sure every row leading entry is 1 and then use E3 to make sure entries above row. leading entries are zero. $\square$.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{E2} \underset{(a)}{\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}} \xrightarrow[R1 \leftarrow R1 - 2R2]{E3} \underset{(b)!}{\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

DONE!

E.g.

(a) $\begin{bmatrix} 1 & 3 & 7 \\ 2 & 2 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 7 \\ 0 & -4 & -8 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 7 \\ 0 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

E.F. R.E.F.

(b) $\begin{bmatrix} 1 & 3 & 7 \\ 2 & 6 & 14 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 7 \\ 0 & 0 & 0 \end{bmatrix}$

E.F. and actually REF.

(c). $\begin{bmatrix} 0 & 3 & 7 \\ 2 & 6 & 10 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 6 & 10 \\ 0 & 3 & 7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & \frac{7}{3} \end{bmatrix} \xrightarrow[R1-3R2]{R1 \uparrow} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & \frac{7}{3} \end{bmatrix}$

E.F. R.E.F.

(d) $\begin{bmatrix} 1 & 3 & 7 \\ 2 & 6 & 10 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 7 \\ 0 & 0 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 7 \\ 0 & 0 & 1 \end{bmatrix}$

E.F. R.E.F.

2. Solns of L.E.S. from Echelon forms

E.g. (2). $\begin{cases} x+2y+3z=4 \\ 5x+6y+7z=8 \\ 9x+10y+11z=12 \end{cases} \longrightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$

Gaussian elimination

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{cases} x \quad\; -z = -2 \\ \quad y+2z = 3 \\ 0x+0y+0z=0. \end{cases}$$

Duh!

inf. many solns.

next time: get this into "parametric vector form".

$$\text{soln set} = \left\{ (x,y,z) \;\middle|\; \begin{array}{l} x = z-2 \\ y = 3-2z \end{array},\ z \in \mathbb{R} \right\} = \left\{ (z-2,\ 3-2z,\ z) \;\middle|\; z \in \mathbb{R} \right\}.$$

(2). (*) $\begin{cases} 3z = 9 \\ 2x - z = 5 \\ 2y + z = 1 \end{cases} \longrightarrow \begin{bmatrix} 0 & 0 & 3 & 9 \\ 2 & 0 & -1 & 5 \\ 0 & 2 & 1 & 1 \end{bmatrix} =: A.$

Ex: Find a reduced ech. form of A. Then find the soln set of (*).

Next time: consistency/soln sets from echelon forms.