Math 2130. Lecture 29.

03. 29. 2021.

Midterm II reminder: available on Canvas 5:59 – 11:59 pm on Mon. Apr. 05. covers roughly Ch 2 – Ch 4. review posted today.

Last time: homework discussion

Today: 4.7. Change of basis. → will be on Midterm II. → will be very useful for Ch 5.

Motivation: to relate different bases of a v.s. more precisely, given a v.s. V and two bases $\mathcal{B}$ and $\mathcal{C}$ of V, for every $v \in V$ we have two coordinate vectors $[v]_{\mathcal{B}}$ and $[v]_{\mathcal{C}}$.

Q: How are $[v]_{\mathcal{B}}$ and $[v]_{\mathcal{C}}$ related?

Example. $V = \mathbb{R}^2$, $v = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$, $B = \left(\underset{b_1}{\begin{bmatrix} 0 \\ -1 \end{bmatrix}}, \underset{b_2}{\begin{bmatrix} 3 \\ 5 \end{bmatrix}} \right)$, $C = \left(\underset{c_1}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}, \underset{c_2}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} \right)$

It's easy to check that both B and C are bases of V, so $[v]_B$ and $[v]_C$ make sense. Note that: (not trivial)

(a) $v = \begin{bmatrix} 3 \\ 7 \end{bmatrix} = -2 \cdot \begin{bmatrix} 0 \\ -1 \end{bmatrix} + 1 \cdot \begin{bmatrix} 3 \\ 5 \end{bmatrix} = -2b_1 + b_2$, so $[v]_B = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

(b) $v = \begin{bmatrix} 3 \\ 7 \end{bmatrix} = 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \cdot c_1 + 3 \cdot c_2$, so $[v]_C = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

(c). $b_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -c_1$, so $[b_1]_C = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$;

$b_2 = \begin{bmatrix} 3 \\ 5 \end{bmatrix} = -\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = -c_1 + 3c_2$, so $[b_2]_C = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

so, $P := \left[[b_1]_C \,\vdots\, [b_2]_C \right] = \begin{bmatrix} -1 & -1 \\ 0 & 3 \end{bmatrix}$.

point: $\boxed{[v]_C = P[v]_B}$

A derivation relating $[v]_B$ and $[v]_C$: $\left([v]_C = \left[[b_1]_C \,\vdots\, [b_2]_C \right] [v]_B \right.$

Since the map $[\ ]_C : V \to \mathbb{R}^n$, $v \mapsto [v]_C$ is linear, we have

$$[v]_C = [-2b_1 + 1b_2]_C = -2[b_1]_C + 1[b_2]_C$$

$$= \left[[b_1]_C \;\vdots\; [b_2]_C \right] \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$= \left[[b_1]_C \;\vdots\; [b_2] \right] [v]_B .$$

The above argument can be generalized to prove our main theorem:

Thm Let $\mathcal{B} = \{b_1, \ldots, b_p\}$ and $\mathcal{C} = \{c_1, \ldots, c_p\}$ be two bases of a vec. space V. Then there is a unique matrix $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$, called the change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$, s.t.

$$[v]_{\mathcal{C}} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{P} [v]_{\mathcal{B}} \quad \text{for all } v \in V.$$

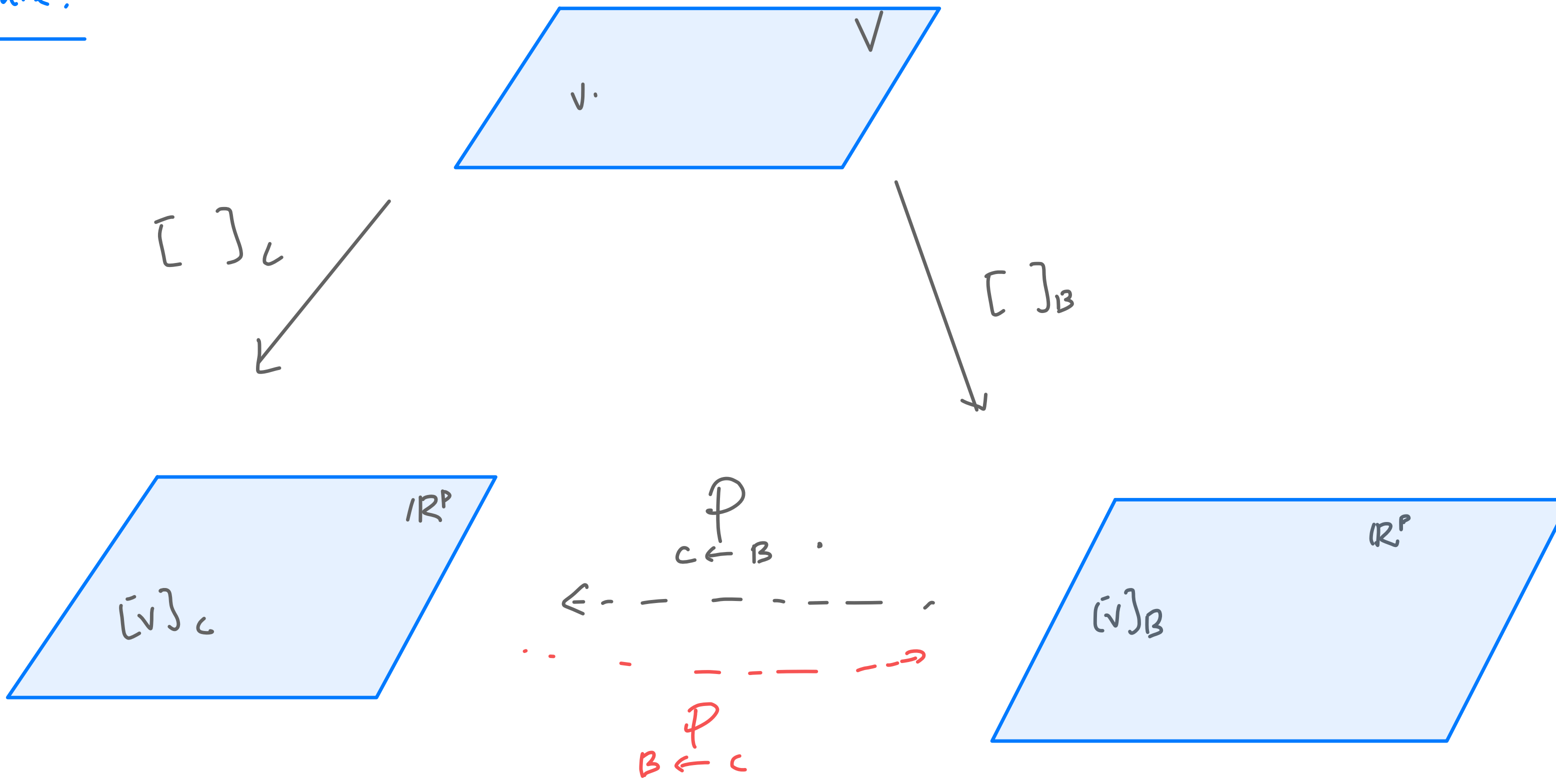
This matrix is given by

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \left[[b_1]_{\mathcal{C}} \;\middle|\; \cdots \;\middle|\; [b_p]_{\mathcal{C}} \right].$$

Moreover, we have

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \left(\underset{\mathcal{B} \leftarrow \mathcal{C}}{P} \right)^{-1}.$$

Picture:



" $P_{C\leftarrow B} = (P_{B\leftarrow C})^{-1}$ " :

$$P_{B\leftarrow C} \cdot P_{C\leftarrow B} = P_{B\leftarrow B} = I_p$$

$$P_{C\leftarrow B} \cdot P_{B\leftarrow C} = P_{C\leftarrow C} = I_p .$$

Example. Suppose that $\mathcal{A} = (a_1, a_2, a_3)$ and $\mathcal{B} = (b_1, b_2, b_3)$ are two bases of a vec. space V s.t. $a_1 = 4b_1 - b_2$, $a_2 = -b_1 + b_2 + b_3$, $a_3 = b_2 - 2b_3$.

(i). Find $\underset{\mathcal{A} \leftarrow \mathcal{B}}{P}$ and $\underset{\mathcal{B} \leftarrow \mathcal{A}}{P}$.

(ii). Find $[x]_{\mathcal{B}}$ for $x = 3a_1 + 4a_2 + a_3$.

// Method 2

$$3(4b_1 - b_2) + 4(-b_1 + b_2 + b_3) + (b_2 - 2b_3)$$
$$\parallel$$
$$8b_1 + 2b_2 + 2b_3$$
$$\Downarrow$$
$$[x]_{\mathcal{B}} = \begin{bmatrix} 8 \\ 2 \\ 2 \end{bmatrix}.$$

Soln: (i). We first find $\underset{\mathcal{B} \leftarrow \mathcal{A}}{P}$. By assumption, we have

$$[a_1]_{\mathcal{B}} = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}, \quad [a_2]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \quad [a_3]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}.$$

So $\underset{B \leftarrow A}{P} = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}$.

To find $\underset{A \leftarrow B}{P}$, we use the fact that $\underset{A \leftarrow B}{P} = \underset{B \leftarrow A}{P}^{-1}$.

$$\left[\begin{array}{ccc|ccc} 4 & -1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \end{array}\right] \sim \cdots\cdots$$

Ex.

(ii). Method 1.

$$[x]_B = \underset{B \leftarrow A}{P} [x]_A = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ 2 \end{bmatrix}. \quad \square$$

(i.e., $x = 8b_1 + 2b_2 + 2b_3$.)

E.g. Find $\underset{C \leftarrow B}{P}$ for the bases $B = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ and $C = \left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$ of $\mathbb{R}^2$.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad b_1 \qquad b_2 \qquad\qquad\qquad c_1 \qquad c_2$

The underlined vectors imply that $\underset{C \leftarrow B}{P} = \begin{bmatrix} -\frac{5}{2} & -\frac{3}{2} \\ 1 & 1 \end{bmatrix}$.

Soln: We need to have $[b_1]_C$ and $[b_2]_C$:

- finding $[b_1]_C = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$. $x_1 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + y_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, i.e., $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

$$\rightsquigarrow \left[\begin{array}{cc:c} 2 & 3 & -2 \\ 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cc:c} 1 & \frac{3}{2} & -1 \\ 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cc:c} 1 & 0 & -\frac{5}{2} \\ 0 & 1 & 1 \end{array}\right] \Rightarrow [b_1]_C = \underline{\begin{bmatrix} -\frac{5}{2} \\ 1 \end{bmatrix}}.$$

- finding $[b_2]_C = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$, . . . — — — — $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$$\to \left[\begin{array}{cc:c} 2 & 3 & 0 \\ 0 & 1 & 1 \end{array}\right] \to \begin{bmatrix} 1 & \frac{3}{2} & 0 \\ 0 & 1 & 1 \end{bmatrix} \to \left[\begin{array}{cc:c} 1 & 0 & -\frac{3}{2} \\ 0 & 1 & 1 \end{array}\right] \Rightarrow [b_2]_C = \underline{\begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix}}.$$

Next time: algorithm and more examples for computing transition matrices.