Math 2130. Lecture 27.

Midterm II: Monday, April 05.

03. 24. 2021.

Last time:

- Axiomatic def. of abstract vector space. e.g. $\mathbb{P}_n$
- Generalization of familiar notions: subspace, lin ind, span, ⋯.
- Generalization of familiar facts: "Spans are subspaces", ...

Today:

- Connecting abstract vec. spaces to $\mathbb{R}^n$ via coordinate mappings.

  e.g. $\mathbb{P}_3 \xrightarrow{\sim} \mathbb{R}^4$

- "Row rank = Col rank".

1. Coordinate mappings

e.g. $V = P_3$, $B = \{1, t, t^2, t^3\}$

Let V be a vector space and let $B = \{v_1, v_2, \ldots, v_k\}$ be a basis of V.

Thm 1. (Coordinate vectors) For every $v \in V$, there are unique scalars $c_1, \ldots, c_k \in \mathbb{R}$

s.t. $v = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$. (We denote the vector $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$ by $[v]_B$.)

Pf: Existence: B spans V. Uniqueness: B is lin ind. (See Lecture 21. 03/08.)

Thm 2. (Coordinate mappings) The map $[\ ]_B : V \to \mathbb{R}^k$, $v \mapsto [v]_B$ is

(1) linear $\left(\begin{array}{ll} T(v+w) = T(v) + T(w) & \forall v, w \in V \\ T(cv) = cT(v) & \forall v \in V, c \in \mathbb{R} \end{array} \right)$

and (2) bijective.

Pf: (1) Let $v, w \in V$. Then by Thm, $v \overset{(a)}{=} c_1 v_1 + \cdots + c_k v_k$, $w \overset{(b)}{=} d_1 v_1 + \cdots + d_k w_k$

for some unique scalars $c_1, \cdots, c_k, d_1, \cdots, d_k \in \mathbb{R}$.

$$T(v+w) = T\left((c_1+d_1)v_1 + \cdots + (c_k+d_k)v_k \right) = \begin{bmatrix} c_1+d_1 \\ \vdots \\ c_k+d_k \end{bmatrix}$$

and $T(v)+T(w) \overset{(a)}{=} \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix} + \begin{bmatrix} d_1 \\ \vdots \\ d_k \end{bmatrix} = \begin{bmatrix} c_1+d_1 \\ \vdots \\ c_k+d_k \end{bmatrix}$, so $T(v+w) = T(v)+T(w)$.

Let $c \in \mathbb{R}$. then $T(cv) = T\left(cc_1 v_1 + \cdots + cc_k v_k \right) = \begin{bmatrix} cc_1 \\ \vdots \\ cc_k \end{bmatrix}$

$$cT(v) = c \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix} = \begin{bmatrix} cc_1 \\ \vdots \\ cc_k \end{bmatrix}$$, so $T(cv) = cT(v)$.

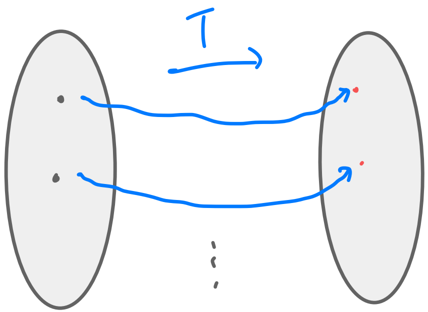
It follows that T is linear.

(2). We need to show that T is injective and surj:

inj: If $T(v) = T(w) = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}$, then $v = w = c_1 v_1 + \cdots + c_k v_k$, so T is inj.

surj: $\forall \vec{x} = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix} \in \mathbb{R}^k$, we have $T(c_1 v_1 + \cdots + c_k v_k) = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}$, so T is surj. $\square$

Point:



"linear isomorphism"

Since T is linear and bijective, it sets up a "dictionary" between (the elts of) V and $\mathbb{R}^k$, allowing us to translate problems about V into ones about $\mathbb{R}^k$.

e.g. $P_3 \longrightarrow \mathbb{R}^4$, $f = 1 - 2t + \pi t^2 - 3t^3 \longleftrightarrow \begin{bmatrix} 1 \\ -2 \\ \pi \\ -3 \end{bmatrix} = [f]_{\{1, t, t^2, t^3\}}$

<u>Key fact</u>: Let V be a vector space and let B be a basis for V. and let $S = \{u_1, \ldots, u_k\}$ be a subset of V. Then

$S = \{u_1, \ldots, u_k\}$ satisfies some property ("being lin ind", "spanning V", ...)

iff $S' = \{[u_1]_B, \ldots, [u_k]_B\}$ satisfies the same property in $\mathbb{R}^k$.

Eg: $V = P_2$. Is the set $S = \{\underbrace{1-t}_{v_1}, \underbrace{2+t^2}_{v_2}, \underbrace{3}_{v_3}, \underbrace{4t}_{v_4}\}$ a spanning set of V? Is S lin ind? Is S a basis of V?

<u>Soln</u>: Consider the standard basis $\{1, t, t^2\}$ for V. Then the map

$[\ ]_B : V \to \mathbb{R}^3$, $v \mapsto [v]_B$ sets up a linear iso between V and $\mathbb{R}^3$.

We have $x_1 := [v_1]_B = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $x_2 := [v_2]_B = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, $x_3 := [v_3]_B = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$, $x_4 := [v_4]_B = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}$.

"Does S span V?" $\longleftrightarrow$ "Does $S' := \left\{ x_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, x_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, x_3 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, x_4 = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} \right\}$

span $\mathbb{R}^3$?"

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ -1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 3 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 \end{bmatrix}$$ EF.

no zero row, so they do span $\mathbb{R}^3$.

So, S spans V.

"Is S lin ind (in V)?" $\longleftrightarrow$ "Is S' lin ind in $\mathbb{R}^3$?"

"Is S a basis for V?"

No, since it's not lin. ind.

No, because a set of four elts cannot be lin ind in $\mathbb{R}^3$ (since $4>3$).

$\square$

2. Row rank = Column rank (Section 4.6)

We discuss Section 4.6, which revisits many notions in $\mathbb{R}^n$ (but from a more abstract perspective). The only new topic is the following:

Def: (Row space) Let A be an $m \times n$ matrix $A = \begin{bmatrix} r_1 \\ \hline \vdots \\ \hline r_m \end{bmatrix}$ (e.g. $A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 2 \end{bmatrix}$)

We define the row space of A to be the span of the rows $r_1^T, \dots, r_m^T$ in $\mathbb{R}^n$ and denote it by $\text{Row}(A)$. ($\text{Row}(A) = \text{Span}\left\{ \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \right\} \subseteq \mathbb{R}^3$.)

Goals:

- Find a basis for $\text{Row}(A)$.
- Find the dimension of $\text{Row}(A)$

i.e. the row rank of A.

Def: The row rank of A is $\dim(\text{Row}(A))$.

Thm: Row equivalent matrices have the same row span.

E.g. $\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \xrightarrow{\sim} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \Rightarrow \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}\right\} = \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix}\right\}$

Reason: Elementary row operations don't change row spans.

Corollary 1. A basis for Row(A) is given by the pivot rows in EF(A).

E.g. $\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}}_{EF} \Rightarrow$ Row A has a basis given by $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix}\right\}$

Note: In general we must pick the pivot rows in EF(A), not A.

Contrast this with the bases of column spaces. March 10.

Example. $A = \begin{bmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{bmatrix} \xrightarrow[\sim]{\text{assume}} \begin{bmatrix} 1 & 0 & -1 & 5 \\ 0 & -2 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\Downarrow$

$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 5 \\ -6 \end{bmatrix} \right\}$ is a basis of $\text{Row}(A)$.

Corollary 2: Recall that the rank of a matrix A is the dimension of $\text{Col}(A)$. (Sometimes we also call this rank the column rank.) The row rank of A equals the (col) rank of A.

Pf: $\text{Row Rank}(A) = \dim(\text{Row}(A)) = |\text{pivot rows in } A| = \#\text{ pivots in } A$

$= |\text{pivot cols in } A| = \text{Rank}(A)$. $(\text{Col Rank}(A))$