Math 2130. Lecture 26. 03.22.2021.

Last time: · Applications of determinants, det and area/volume.

Done with Ch3.

Today: · start Ch4. (Abstract) vector spaces

Theme: An arbitrary abstract v.s. may not be $\mathbb{R}^n$, but will behave in a similar way to $\mathbb{R}^n$.

Specifically: – def & examples of vec. spaces.

– generalizations of familiar notions: linear comb, spans, subspace, linear maps, kernels, images, basis, dim, ...

1. Definition of vector spaces.

Def: A (real) vector space, or a vector space over $\mathbb{R}$, is the data of a triple $(V, +, \cdot)$ where V is a nonempty set, $+$ is a map $+: V \times V \to V$, $(v, w) \mapsto v+w$ called addition, and $\cdot$ is a map $\cdot: \mathbb{R} \times V \to V$, $(c, v) \mapsto c \cdot v$ called scalar multiplication which satisfy the following properties:

(1). $u+v = v+u \quad \forall u, v \in V$ (2). $(u+v)+w = u+(v+w) \quad \forall u, v, w \in V$.

(3). V has a special elt 0 called zero s.t. $u+0 = u = 0+u \quad \forall u \in V$.

(4). $\forall u \in V, \exists v \in V$ s.t. $u+v=0$. (5). $c(u+v) = cu + cv \quad \forall c \in \mathbb{R}, u, v \in V$.

(6). $(c+d) \cdot v = c \cdot v + d \cdot v \quad \forall c, d \in \mathbb{R}, v \in V$ (7). $(cd) \cdot v = c \cdot (d \cdot v) \quad \forall c, d \in \mathbb{R}, v \in V$

(8). $1 \cdot u = u \quad \forall u \in V$. We often refer to V as a vector space too. $\square$

2. New examples of vector spaces.

Apart from $\mathbb{R}^n$, there are many other types of vector spaces:

Eg. (polynomial spaces) Let $n \geq 0$. Let P_n be the set of all polynomials (with real coefficients) of degree at most n, i.e.,

$$P_n = \{ a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n \mid a_0, a_1, \ldots, a_n \in \mathbb{R} \}$$

Then P_n forms a vector space under the usual addition and scalar mult:

e.g. $n=4$, $(2t - t^2) + (1 - 2t + 3t^2 + 4t^3 + 5t^4) = 1 + 2t^2 + 4t^3 + 5t^4$.

$(2t - t^2) + (-2t + t^2) = \underline{0} \rightarrow$ the zero vector,

$0 + 0 \cdot t + \cdots + 0 \cdot t^n$.

$3 \cdot (2t - t^2) = 6t - 3t^2$.

Eg. (function spaces) The set V of all functions $f : \mathbb{R} \to \mathbb{R}$
vector in V.

forms a v.s. under function addition and scaling.

eg. $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto f(x) = \sin x$.

$g : \mathbb{R} \to \mathbb{R}$, $x \mapsto g(x) = 2x$

$f+g$: the function with $(f+g)(x) = f(x) + g(x)$ $\forall x \in \mathbb{R}$.

cf : - - - - . . $(cf)(x) = c \cdot f(x)$ $\forall x \in \mathbb{R}$.

$c \in \mathbb{R}$

The zero function : $0 : \mathbb{R} \to \mathbb{R}$ with $0(x) = 0$ $\forall x \in \mathbb{R}$.

3. Familiar notions, revisited.

Let $(V, +, \cdot)$ be a vec. space.

(1) linear comb. Since addition and scalar multiples make sense, linear comb. make sense: for elts $v_1, v_2, \dots, v_k \in V$, we define a linear combination of them to be an elt of the form $v = c_1 v_1 + c_2 v_2 + \dots + c_k v_k$ for some $c_1, \dots, c_k \in \mathbb{R}$

(2). spans: given $v_1, \dots, v_k \in V$, we define their span to be the set of all their lin comb.

$$\operatorname{Span}(\{v_1, \dots, v_k\}) = \{ c_1 v_1 + \dots + c_k v_k \mid c_1, \dots, c_k \in \mathbb{R} \}.$$

(3). linear independence. We say a set $S = \{v_1, \dots, v_k\}$ is <u>linearly independent</u> if the only soln to the equation

$$c_1 v_1 + \dots + c_k v_k = 0$$

is $c_1 = c_2 = \dots = c_k = 0$.

(4) linear maps, kernel, image, injectivity, surjectivity. Let W be a v.s.

A <u>linear map</u> from V to W is a function $f: V \to W$ s.t.

(a) $f(u+v) = f(u) + f(v)$ $\forall u, v \in V$ and (b) $f(cu) = c f(u)$ $\forall u \in V, c \in \mathbb{R}$.

Given a linear map $f: V \to W$, the <u>kernel</u> of f is the set

$\ker f = \{v \in V : f(v) = 0\}$ and its <u>image</u> is the set $\operatorname{Im}(f) = \{f(v) : v \in V\}$;

We say f is <u>injective</u> if $\ker f = \{0\}$ and <u>surjective</u> if $\operatorname{Im} f = W$.

(5). **Subspace** A <u>subspace</u> of V is a subset U of V s.t.

(a) $0_V \in U$.

(b) $\forall u, v \in U$, $u+v \in U$

(c) $\forall u \in U$, $c \in \mathbb{R}$, $c \cdot u \in U$.

(6) **Basis**. A <u>basis</u> of a subspace U of V is a subset B of U s.t.

(i) B is lin ind ;

(ii) $\mathrm{Span}(B) = U$.

4. Familiar facts, generalized

Many results about $\mathbb{R}^n$ carry over to abstract vector spaces with the same proof.

E.g. (Spans are subspaces) Let V be a vec. space, and let $S = \{v_1, \dots, v_k\}$ be a subset of V. Then $\mathrm{Span}(S)$ is a subspace of V.

Pf: (same as before; see 03.05) (a). $0 = 0 \cdot v_1 + \dots + 0 \cdot v_k \in \mathrm{Span}(S)$.

(b) Take $u, v \in \mathrm{Span}(S)$. Then $u = a_1 v_1 + \dots + a_k v_k$, $v = b_1 v_1 + \dots + b_k v_k$ for some $a_i, b_i \in \mathbb{R}$.

But then $u + v = \dots = (a_1 + b_1) v_1 + \dots + (a_k + b_k) v_k \in \mathrm{Span}(S)$.

(c). Ex.

By (a), (b), (c), $\mathrm{Span}(S)$ is a subspace of V.

Eg. Bases: The following holds for any vec. space V:

(1). Every two bases of V have the same cardinality/size.

This common cardinality is called the <u>dimension</u> of V.

(2). Suppose $\dim(V) = k$ for some $k \geq 0$, i.e., V has a basis with k vectors.

Then for any subspace S of V, we have S is a basis of V if any two of the following three conditions hold:

(i). $|S| = k$ (ii). S is lin. ind. (iii) S spans V.

E.g. Let $V = P_4$.

(1). Prove that $P_3 = \{ a_0 + a_1 t + a_2 t^2 + a_3 t^3 + 0 \cdot t^4 \mid a_0, a_1, a_2, a_3 \in \mathbb{R} \}$ is a subspace of P_4.

(2). Prove that $S = \{ 2, t-1, 3t^2, t^3 \}$ is basis for P_3.

Pf: (1). Note that $P_3 = \text{Span}\{1, t, t^2, t^3\}$. Since spans are subspaces, P_3 is a subspace.

(2). Note that $B = \{1, t, t^2, t^3\}$ is a basis for P_3: (i). it's linearly ind.: $a + bt + ct^2 + dt^2 = 0 \Rightarrow a = b = c = d = 0$; (ii). it spans P_3, by (1). So $\dim P_3$ is 4. Now S has 4 vectors and we can check that S is lin. ind. (Ex.). It follows that S is a basis for P_3.

Next time: we'll turn problems like the previous one to problems about vec. spaces of the form $\mathbb{R}^n$ via so-called coordinate mappings.

e.g. P_4. $\quad B = \{1, t, t^2, t^3, t^4\}$.

$$f = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 \longmapsto \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = [f]_B$$