Math 2130. Lecture 25.

03. 19. 2021.

Last time:

- more examples of det computations
- properties of det. (w.r.t. row operations.)

$\Downarrow$

application: proof of "Thm 1": $\det A \neq 0 \iff A$ is inv.

(A square)

Today:

- more applications of det.
  - determining inv. of matrix products.
  - computing matrix inverses.
  - solving matrix equations.
  - computing areas/volumes.

} can already do with row reductions.

1. Invertibility of matrix products

Recall Thm 2: Let A, B be $n\times n$ matrices. Then $\det(AB) = \det(A)\det(B)$.

Corollary: Let A, B be $n\times n$ matrices. Then AB is inv. iff A and B are both inv. (In fact, let $A_1, A_2, \ldots, A_k$ be $n\times n$ matrices. Then $A_1 A_2 \cdots A_k$ is inv iff $A_1, A_2, \ldots, A_k$ are all invertible.)

Pf: $(A_1 A_2 \cdots A_k)$ is inv $\overset{\text{Thm 1}}{\Longleftrightarrow} \det(A_1 A_2 \cdots A_k) \neq 0$

$\overset{\text{Thm 2}}{\Longleftrightarrow} \det(A_1) \cdots \det(A_k) \neq 0$

$\Longleftrightarrow \det(A_1) \neq 0,\ \det(A_2) \neq 0,\ \ldots,\ \det(A_k) \neq 0$

$\Longleftrightarrow A_1, A_2, \ldots, A_k$ are all inv. $\square$

*2. Finding matrix inverses

Let A be an $n \times n$ matrix.

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ inv $\Rightarrow A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Fact: If $\det A \neq 0$ (i.e., if A is inv.), then

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} & & \\ & x_{ij} & \\ & & \end{bmatrix}$$

entry in ith row, jth col

where $x_{ij} = (-1)^{i+j} \det A_{ji}$

submatrix of A obtained by deleting the jth row, ith col.

Eg. **2×2:**

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} (-1)^{1+1}d & (-1)^{1+2}b \\ (-1)^{2+1}c & (-1)^{2+2}a \end{bmatrix} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

3* Solving matrix equations.

Consider the matrix eq. $\underset{n\times n}{A}x = \underset{\mathbb{R}^n}{b}$. (✱) $\left(A = [v_1 | \cdots | v_n]\right)$

(a) Recall that if A is inv, then (✱) has a unique soln, namely $A^{-1}b$, so since we can compute A^{-1} via determinants, we can solve (✱) via det.

b) There is also another, more direct way to solve (✱) via determinants:

Fact: If A is inv, the soln to $Ax = b$ is given by the vector $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

where $$x_i = \frac{\det(A_i(b))}{\det A}.$$

$$A_i(b) = [v_1 | \cdots | v_{i-1} | \underset{\text{ith col}}{b} | v_{i+1} | \cdots | v_n].$$

E.g. $\begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$. $\Rightarrow$ $x_1 = \dfrac{\begin{vmatrix} 6 & -2 \\ 8 & 4 \end{vmatrix}}{2} = \dfrac{40}{2} = 20.$

det: 2

$x_2 = \dfrac{\begin{vmatrix} 3 & 6 \\ -5 & 8 \end{vmatrix}}{2} = \dfrac{54}{2} = 27.$

check: ✓.

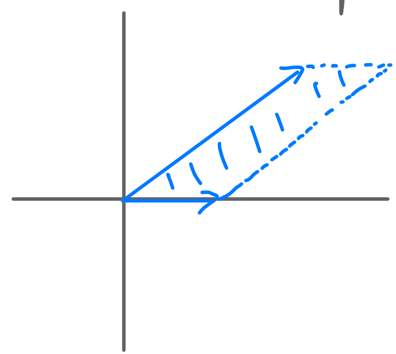
$x_1 = 20$, $x_2 = 27$

4. Determinants as area / volume

(a) Det of 2×2 matrices vs area:

Thm 1. (Thm 3.3.9.) Given a 2×2 matrix $A = \left[v_1 \mid v_2 \right]$, the area of the parallelogram determined by (with sides) v_1 and v_2 equals $|\det A|$.

E.g. $A = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$



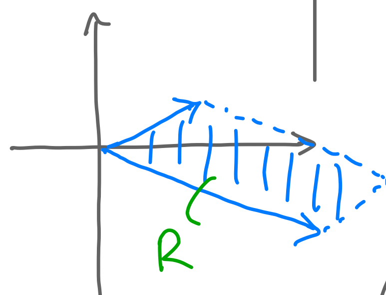
from grade school:

$$\text{Area}(\square) = \text{base} \cdot \text{height} = 2 \times 3 = 6$$

$$\| \checkmark$$

$$|\det A| = |2 \times 3 - 4 \cdot 0| = 6$$

E.g. $B = \begin{bmatrix} 2 & 5 \\ 1 & -2 \end{bmatrix}$



$$\text{Area}(R) = |\det B| = |-9| = 9$$

Thm 2. (Thm 3.3.10) Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map. If P is a parallelogram in $\mathbb{R}^2$, then so is $T(P)$, and

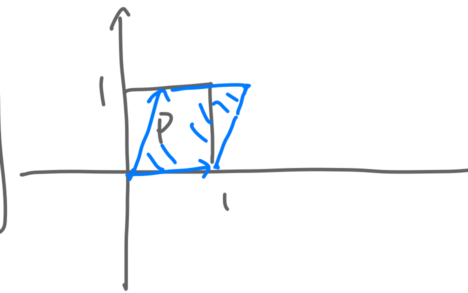
(can be generalized to other shapes/regions like a disk)

$$\text{Area}\,(T(P)) = \left| \det A_T \right| \cdot \text{Area}\,(P) \quad (*)$$

where A_T is the standard matrix of T.

E.g. $P =$ the unit square, $\underline{\underline{T}}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \underbrace{\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}}_{A_T} \begin{bmatrix} x \\ y \end{bmatrix}$

horizontal shear

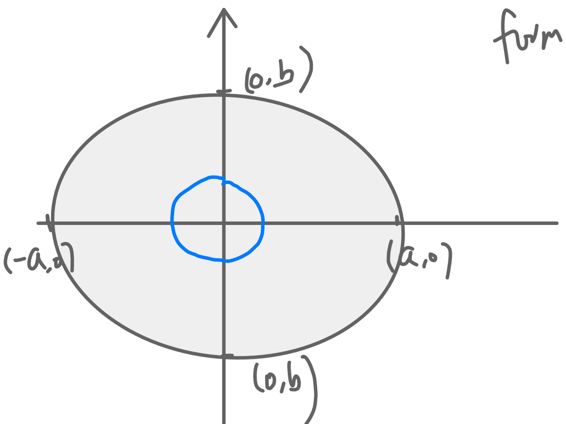


$\text{Area}(T(P)) = 1$ by geometry

$\text{Area}(P) = 1 \times 1 = 1$, $\left|\det A_T\right| = \left| 1 \right| = 1$. so $(*)$ does hold.

E.g. Let a, b be positive numbers. The points (x,y) satisfying the eq.

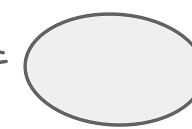
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



form an ellipse enclosing a region in $\mathbb{R}^2$.

Fact: We may take the unit circle and stretch it horizontally by a factor of a, vertically by b to obtain the enclosed region.

$A_T = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$

$\left(T : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} ax \\ by \end{bmatrix} \right.$ satisfies $T($  $) =$  $\Big)$

Thus, Area() = Area() $\cdot \left| \det A_T \right| = \pi \cdot 1^2 \cdot ab = \underline{\underline{\pi ab}}$

1b). 3×3 dets vs. parallelopiped:

Thm 1: (3.3.9.) Given a 3×3 matrix $A=[v_1|v_2|v_3]$, the volume of the parallelopiped determined by v_1, v_2, v_3 is $|\det A|$.

Thm 2. (3.3.10.) Given a parallelopiped P in $\mathbb{R}^3$ and a linear map $T: \mathbb{R}^3 \to \mathbb{R}^3$, then

can be generalize to other "nice" regions.

$$\mathrm{Vol}(T(P)) = |\det A_T| \, \mathrm{Vol}(P)$$

where A_T is the standard matrix of T.