Math 2130. Lecture 24. 03.17.2021.

Last time: · Coordinate systems: given subspace $V \subseteq \mathbb{R}^n$ and a basis $B = \{v_1, \cdots, v_k\}$ of V, we have a map $V \to \mathbb{R}^k$, $v \mapsto \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} =: [v]_B$

where $c_1, c_2, \ldots, c_k$ are the unique scalars s.t. $v = c_1 v_1 + \cdots + c_k v_k = \begin{bmatrix} v_1 & \cdots & v_k \end{bmatrix} [v]_B$

· Determinants of square matrices. 1×1 : $|a| = a$ $\quad$ 2×2 : $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$n\times n$, $n \geq 3$: cofactor expansion along any row or column.

Key : take a sum with n summands, each summand containing three factors (matrix entry, sign, det of a smaller matrix)

· Theorems: 1. For every square matrix A, $\det A \neq 0 \Leftrightarrow A$ is inv.

2. For $n\times n$ matrices A, B, $\det(AB) = \det A \det B$.

Today: Properties and applications of determinants.

Example. Determinant of a 4x4 matrix.

$$
\begin{vmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 0 & 5 & 1 & -6 \end{vmatrix}
= \left((-1)^{1+3} \cdot 2 \cdot \begin{vmatrix} 0 & 3 & -4 \\ -5 & -8 & 3 \\ 0 & 5 & -6 \end{vmatrix} + \overset{0}{\cancel{\cdot 0 \cdot}} + \underset{0}{\cancel{\cdot 0 \cdot}} + (-1)^{4+3} \cdot 1 \cdot \begin{vmatrix} 5 & -7 & 2 \\ 0 & 3 & -4 \\ -5 & -8 & 3 \end{vmatrix} \right)
$$

$\checkmark$

$$
= 1 \cdot 2 \cdot \left((-1)^{2+1} \cdot (-5) \cdot \begin{vmatrix} 3 & -4 \\ 5 & -6 \end{vmatrix} \right) + (-1) \cdot 1 \cdot \left((-1)^{2+2} \cdot 3 \cdot \begin{vmatrix} 5 & 2 \\ -5 & 3 \end{vmatrix} + (-1)^{2+3} \cdot (-4) \cdot \begin{vmatrix} 5 & -7 \\ -5 & -8 \end{vmatrix} \right)
$$

$$
= 2 \left(5 \cdot 2 \right) - \left(3 \cdot (25) + 4 \cdot (-75) \right)
$$

$$
= 20 - 75 + 300 = 245
$$

$\neq 0 \Rightarrow$ the matrix is inv.

A shortcut (or not) for computing 3×3 det :

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh$$

$$- ceg - bdi - ahf .$$

1. Properties of Determinants

Let A be an $n \times n$ matrix. (e.g. $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$ or $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$)

Facts. (Effect of row operations on determinants)

(1) (Effect of scaling) **Fact:** If we scale one row of a square matrix A to obtain a square matrix B by a scalar c, then

$$\det B = c \det A$$

E.g. $A = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \xrightarrow{R_2 \to c \cdot R_2} B = \begin{bmatrix} x & y \\ cz & cw \end{bmatrix}$: $\det B = x(cw) - y \cdot (cz)$

$= c(xw - yz) = c \det A.$

$A: 2\times 2 \Rightarrow \det(-A) = (-1)^2 \cdot \det A = \det A$; $A: n\times n \Rightarrow \det(-A) = (-1)^n \det A.$

(2). Effect of interchange

Fact: If we swap two rows in a square matrix A to obtain a matrix B, then $\det B = -\det A$.

e.g. $\det \underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{A} = 1\cdot 4 - 2\cdot 3$ $\qquad \det \underbrace{\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}}_{B} = 3\cdot 2 - 1\cdot 4 = -\det B$

Ex: Convince yourself that $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -\begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix}$.

Corollary: If a square matrix A has a row (Row i) which is the same of another row (Row j), then it has det. 0.

Pf: $\boxed{\det A} = -\det(A, \text{ with Row } i, \text{ Row } j \text{ swapped}) = -\boxed{\det(A)} \Rightarrow \det A = 0$.

Corollary': If A is a square matrix where one row is a scalar multiple of another, then $\det A = 0$.

Pf: Ex.

e.g. $\begin{vmatrix} a & b & c \\ d & e & f \\ xa & xb & xc \end{vmatrix} = x \begin{vmatrix} a & b & c \\ d & e & f \\ a & b & c \end{vmatrix} = x \cdot 0 = 0$.

(3) (Effect of replacement) **Fact**: If we add a multiple of a row to a sq. matrix A to another row, then the resulting matrix B has the same det as the original matrix. ($\det B = \det A$).

$\begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 5 & 4 \\ 1 & 0 \end{vmatrix} = \begin{vmatrix} 7 & 7 \\ 1 & 0 \end{vmatrix}$

e.g. $\underbrace{\begin{vmatrix} a & b & c \\ d-2a & e-2b & f-2c \\ g & h & i \end{vmatrix}}_{B} = \underbrace{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}_{A}$

A related fact: $\begin{vmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{vmatrix} + \begin{vmatrix} r_1' \\ r_2 \\ \vdots \\ r_n \end{vmatrix} = \begin{vmatrix} r_1 + r_1' \\ r_2 \\ \vdots \\ r_n \end{vmatrix}$

see "A linearity property of det" on Page 3.2. of the textbook.

Note: If two square matrices A, B are row equivalent (i.e., B can be obtained from A via a sequence of elt row operations), then $\det A$ and $\det B$ are either both 0 or both nonzero.

An application of the properties: We can now prove Thm 1:

Thm 1: Let A be a square matrix. Then

$$\det A \neq 0 \iff A \text{ is inv.}$$

detects invertibility, may be viewed as an application of det.

Pf: A is inv $\iff \mathrm{REF}(A) = I_n \iff \det(\mathrm{REF}(A)) \neq 0$

But A is row equiv to $\mathrm{REF}(A)$, so $\det(\mathrm{REF}(A)) \neq 0 \iff \det(A) \neq 0$.

The desired conclusion follows. $\square$

Next time, more application/interpretations of det.