Math 2130. Lecture 23.

03. 12. 2021.

Last time:

- Finding bases and dimensions of subspaces of $\mathbb{R}^n$
- The rank-nullity theorem.
  - matrix version: for every $m\times n$ matrix A, we have
    $$\text{Rank}(A) + \text{Nullity}(A) = \#\text{ cols in } A = n$$
  - linear map version: for every linear map $T: \mathbb{R}^n \to \mathbb{R}^m$, we have
    $$\dim(\operatorname{Im} T) + \dim(\ker T) = \dim\left(\begin{matrix}\text{the}\\ \text{domain}\end{matrix}\right) = n.$$

Today:

- Coordinate systems.
- Determinants. (Ch. 3.)

Coordinate systems.

Let V be a subspace of $\mathbb{R}^n$ and let $B = (v_1, v_2, \cdots, v_k)$ be an ordered basis of V.

Recall that every elt $v \in V$ can then be decomposed as a linear comb

$$v \overset{*}{=} c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$$

in a unique way (i.e., $c_1, c_2, \ldots, c_k$ are uniquely determined).

Def: We call the vector $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$ where $c_1, \ldots, c_k$ are the unique coefficients for which (*) holds the <u>coordinate vector of v relative to B</u> and denote it by $[v]_B$.

E.g. $V = \mathbb{R}^2$, $v = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$$\left\{ \begin{array}{l} B_1 = \left(\overset{v_1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}, \overset{v_2}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \right) \Rightarrow v = 3v_1 + 5v_2, \text{ so } [v]_{B_1} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \\ B_2 = \left(\underset{u_1}{\begin{bmatrix} 3 \\ 0 \end{bmatrix}}, \underset{u_2}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \right) \Rightarrow v = \overset{1 \cdot u_1 + 5 \cdot u_2}{?\, u_1 + ??\, u_2}, \text{ so } [v]_{B_1} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}. \end{array} \right.$$

still a basis

Important equality:

$$v \overset{\text{"}}{=} \underbrace{\begin{bmatrix} v_1 & v_2 & \cdots & v_k \end{bmatrix}}_{\text{"}M_B\text{, matrix for } B\text{"}} [v]_B$$

$$v = c_1v_1 + c_2v_2 + \cdots + c_kv_k$$

Two kinds of problems:

(1) Given B (M_B) and $[v]_B$, find v. E.g. $V = \mathbb{R}^3$. $B = \left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 10 \end{bmatrix} \right)$ a basis. If we have $[v]_B = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$, then

$$v = 2\cdot\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 1\cdot\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} - 1\cdot\begin{bmatrix} 7 \\ 8 \\ 10 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 10 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}.$$

(2). Given B and v, find $[v]_B$. E.g. V, B as above. $v = \begin{bmatrix} 7 \\ 1 \\ 1 \end{bmatrix}$. what's $[v]_B$? Need $\vec{x} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ s.t. $c_1\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2\begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix} + c_3\begin{bmatrix} 7 \\ 8 \\ 10 \end{bmatrix} = v$,

i.e., we need to solve $\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 10 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \\ 1 \end{bmatrix}$, i.e., $M_B \cdot \vec{x} = \vec{v}$. ......

2. Determinants

We'll associate a number to each square matrix A called its <u>determinant</u>; we denote it by $\det A$ or $|A|$ ($\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$ instead of $\left|\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right|$)

- Determinants for 1×1 matrices. $A=[a] \to \det A \stackrel{\text{def}}{=} a$

- Determinants for 2×2 matrices.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow \det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

<u>Thm.</u> (an application of determinants) A 2×2 matrix A is invertible if and only if $\det A \neq 0$. If $\det A \neq 0$, then

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Determinants for larger square matrices

Let $A = [\underset{\uparrow}{a_{ij}}]$ $n\times n$. $n \geq 2$

$i, j \in \{1, 2, \cdots, n\}$.

$a_{ij} = A_{ij}$, the entry in A on the ith row, jth col.

Cofactor expansion: We can compute $\det A$ using determinants of smaller matrices via the formula

$$\det A = \sum_{j=1}^{n} (-1)^{1+j} \cdot a_{1j} \cdot \det C_{1j}$$

where C_{1j} is the submatrix obtained from A by removing the 1st row and jth column.

$n=2$ $\quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = (-1)^{1+1} \cdot a \cdot d + (-1)^{1+2} \cdot b \cdot c = ad - bc.$

Eg.

$n=3$.

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{pmatrix} (-1)^{1+1} \cdot 1 \cdot \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} \\ + (-1)^{1+2} \cdot 2 \cdot \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} \\ + (-1)^{1+3} \cdot 3 \cdot \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 1 \cdot (-3) \\ - 2 \cdot (-6) \\ + 3 \cdot (-3) \end{pmatrix} = 0$$

3×3 example:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$\downarrow$

$$C_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

$$C_{12} = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$$

$$C_{13} = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

In fact, we can compute $\det A$ by cofactor expansion along any row or any col of A.

e.g. $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 0 \\ 0 & 5 & 1 \end{bmatrix}$ $\xrightarrow{\text{1st row}}$ $+1 \cdot 1 \cdot 0 + (-1)\, 2 \cdot (-1) + 1 \cdot 1 \cdot (-5) = -3$.

2nd row $(-1)^{2+1}(-1) \cdot (-3) + \underbrace{(-1)^{2+2} \cdot \underline{\underline{0}} \cdot 1 + (-1)^{2+3} \cdot \underline{\underline{0}} \cdot 5}_{\text{certainly } 0\,!} = -3$.

$\Downarrow$

for any fixed i or j.

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} \cdot a_{ij} \cdot \det C_{ij} \quad \text{expansion along Row } i.$$

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} \cdot a_{ij} \cdot \det C_{ij} \quad \text{expansion along Col } j.$$

e.g. This gives six ways to compute $\det A$ when A is 3×3.

Note: Picking a row/col with the most zeros can make the det. computation easier.

Prop: (determinant of triangular matrices)

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & & \\ 0 & 0 & \ddots & \\ \cdots & \cdots & 0 & a_{nn} \end{vmatrix} = a_{11} \cdot a_{22} \cdot \cdots \cdot a_{nn}.$$

all zeros below the diagonal

e.g. $\begin{vmatrix} 3 & 1 & 0 \\ 0 & 2 & -5 \\ 0 & 0 & 1 \end{vmatrix} = +3 \cdot \begin{vmatrix} 2 & -5 \\ 0 & 1 \end{vmatrix} = 3 \cdot 2 \cdot 1.$

more next time!

Main theorems.

Thm 1 (det. and inv.) For every square matrix A, we have

A is invertible if and only if $\det A \neq 0$.

Thm 2. (det. and products.) For every two square matrices A, B of the same shape, we have

$$\det(AB) = \det A \det B.$$