Math 2130. Lecture 22.

03-10-2021.

Last time:

- Null spaces
- Bases of subspaces of $\mathbb{R}^n$
  - (i) is lin ind.
  - (ii) spans the subspace
- Bases of $\mathbb{R}^n$ and null spaces

Today:

- Dimension.
- Bases criteria / computation.
- Rank and Nullity.

revisit

1. Dimension

Let V be a subspace of $\mathbb{R}^n$.

Fact: All bases of V have the same number of elts (size).

Def. (Dimension) The dimension of V is the common cardinality of all its bases. We write the dimension as $\dim(V)$.

E.g. · $V = \mathbb{R}^n$: we saw that any basis of V must have n vectors, so $\dim(\mathbb{R}^n) = n$.

· $V = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\} \subseteq \mathbb{R}^3$: Note that $B = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ is a basis of V: (i) it's lin. ind. since it contains a single nonzero vec. (ii) it spans V by the def of V. It follows that $\dim V = |B| = 1$.

geometrically, this is a line in $\mathbb{R}^3$.

- $V = \operatorname{Span}\left\{ \underset{v_1}{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}, \underset{v_3}{\begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}} \right\} \subseteq \mathbb{R}^3$

<u>Note:</u> The set $B = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis of V:

(i): $\begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \to \begin{array}{cc} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} & \text{EF} \\ \begin{array}{cc} p & p \end{array} & \end{array}$ all cols are pivot, so B is lin ind.

(ii): $v_3 = 2v_1 + 5v_2$, so a typical elt $av_1 + bv_2 + cv_3$ in V

can always be written as $av_1 + bv_2 + c(2v_1 + 5v_2)$

$= (a + 2c)v_1 + (b + 5c)v_2 \in \operatorname{Span} B$

So B spans V.

It follows that B is a basis of V, hence $\dim V = |B| = 2$.

2. Computation of bases and dimensions.

(a) $\mathbb{R}^n$.

We saw that any basis of $\mathbb{R}^n$ must have n vectors

therefore $\mathbb{R}^n$ has dimension n.

From the last lecture, we also see that:

Thm: (Basis criteria for $\mathbb{R}^n$) A set $S \subseteq \mathbb{R}^n$ is a basis of $\mathbb{R}^n$ whenever two of following three conditions hold:

(1) $|S| = n$ (2) S is lin. ind (3). S spans $\mathbb{R}^n$.

Examples: see the last lecture.

(b) Null space / Kernels.

Let A be an $m\times n$ matrix and let $T: \mathbb{R}^n \to \mathbb{R}^m$ be the associated linear map with $T(x) = Ax$ $\forall x \in \mathbb{R}^n$. Then $\operatorname{Ker} T = \operatorname{Null} A = \{x \in \mathbb{R}^n \mid T(x) = Ax = 0\}$, the soln set of the equation $Ax = 0$, which has a p.v.f.

Def (Nullity) The <u>nullity</u> of A is $\operatorname{Nullity}(A) = \dim(\operatorname{Null} A)$.

Thm (basis and dimension of null spaces)

The constant vectors in the p.v.f. of the soln set $\{x \mid Ax = 0\}$ form a basis for $\operatorname{Null} A = \operatorname{Ker} T$. In particular,

$$\operatorname{Nullity}(A) = \#\text{ non-pivot cols of } A = n - \#\text{ pivots in } A.$$

Example in two pages.

(c) Column spaces / Images

Let A be an $m\times n$ matrix and let $T: \mathbb{R}^n \to \mathbb{R}^m$ be the associated linear map with $T(x) = Ax$ $\forall x \in \mathbb{R}^n$. Then $A = [T(e_1)|\cdots|T(e_n)]$ and

$\text{Col}\, A = \text{Span}\{T(e_1), T(e_2), \ldots, T(e_n)\} = \text{Im}\, T$. (e.g. $T(2e_1+3e_2-e_3) = 2T(e_1)+3T(e_2)-T(e_3)$)

Def (Rank) The rank of A is $\dim(\text{Col}\, A)$.

Thm (basis and dimension of column spaces)

The pivot cols of A (not $EF(A)$!) form a basis form $\text{Col}\, A$.

In particular, $\text{rank}\, A = \#$ pivot cols in A.

"Rank-Nullity Thm"

Corollary:
- For any $m\times n$ matrix A, we have $\text{Nullity}(A) + \text{Rank}(A) = \#\text{cols of } A = n$.
- For any linear map $T: \mathbb{R}^n \to \mathbb{R}^m$, we have $\dim(\ker T) + \dim(\text{Im}\, T) = \dim(\text{the domain}) = n$.

Examples. (i) $A = \begin{array}{c} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \\ \begin{matrix} \text{p} & \text{p} & \text{n.p.} & \text{n.p} \end{matrix} \end{array} \longrightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -24 \end{bmatrix} \longrightarrow \begin{array}{c} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} \text{p} & \text{p} & & \end{matrix} \end{array}$ EF.

The first two cols of A are pivot while the last two cols are not,

so $\operatorname{Col}(A)$ has a basis $\left\{ \begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 10 \end{bmatrix} \right\}$ and hence has dimension 2, i.e., $\operatorname{Rank} A = 2$

Also, we have $\operatorname{Nullity} A = \#$ non-pivot cols in $A = 2$.

However, we don't know a basis for $\operatorname{Null} A$ yet. To obtain a basis for $\operatorname{Null} A$, we

reduce $EF(A)$ further to solve $A\vec{x} = 0$.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{array}{c} \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} x & y & z & w \end{matrix} \end{array} \rightarrow \left\{ \vec{x} \in \mathbb{R}^4 \,\middle|\, A\vec{x} = 0 \right\} = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} : \begin{matrix} x - z - 2w = 0 \\ y + z + 3w = 0 \end{matrix} \right\}$$

$$= \left\{ \begin{bmatrix} z + 2w \\ -z - 3w \\ z \\ w \end{bmatrix} : z, w \in \mathbb{R} \right\} = \left\{ z \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} : z, w \in \mathbb{R} \right\} \Longrightarrow$$

The set $B = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $\operatorname{Null}(A)$.

(ii) Find a basis for $\mathrm{Col}\Bigg(\underbrace{\begin{bmatrix} 1 & 3 & 2 \\ 5 & 0 & 0 \\ 2 & 1 & 5 \end{bmatrix}}_{A}\Bigg)$.

Sol:

$$\begin{bmatrix} 1 & 3 & 2 \\ 5 & 0 & 0 \\ 2 & 1 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 2 \\ 0 & -15 & -10 \\ 0 & -5 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 2 \\ 0 & -5 & 1 \\ 0 & -15 & 10 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 2 \\ 0 & -5 & 1 \\ 0 & 0 & 7 \end{bmatrix}$$

It follows that the pivot cols $\left\{ \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix} \right\}$ form a basis of $\mathrm{Col}(A)$.

(iii) Find a basis for $\mathrm{col}\Bigg(\underbrace{\begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & 4 \\ 0 & 1 & 5 \end{bmatrix}}_{B}\Bigg)$. $= \mathrm{Span}\left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix} \right)$

Sdn:

$$\begin{array}{c} \begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & 4 \\ 0 & 1 & 5 \end{bmatrix} \\ \begin{matrix} p & p & \end{matrix} \end{array} \longrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \text{EF}.$$

The pivot cols are the first two.

It follows that $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathrm{Col}(B)$.

3. The Rank-Nullity Thm.

Def. We define the dimension of $\{0\}$ to be 0.

Fact: If we have subspaces U, V in $\mathbb{R}^n$ with $U \subseteq V$, then $\dim U \le \dim V$, with equality holding iff $U = V$.

$\downarrow$

Consequence: Given any subspace $V \subseteq \mathbb{R}^n$. $\dim V \le \dim \mathbb{R}^n = n$, with $\dim V = n$ only if $V = \mathbb{R}^n$. Similarly, $0 \subseteq V$ so $\dim V \ge \dim 0 = 0$, with $\dim V = 0$ only if $V = 0$.

Point: By the Rank-Nullity Thm. for any linear map $T: \mathbb{R}^n \to \mathbb{R}^m$ and its standard matrix, we have

"A has full rank"

(1) T is surj $\iff \operatorname{Im} T = \mathbb{R}^m \iff \dim \operatorname{Im} T = \operatorname{Rank} A = m$

(2) T is inj $\iff \operatorname{Ker} T = 0 \iff \dim \operatorname{Ker} T = \operatorname{Nullity} A = 0$

} Some applications: next time.