Math 2130. Lecture 21.

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Last time: · Example subspaces (of $\mathbb{R}^n$): $\{0\}$, $\mathbb{R}^n$, spans, images, kernels, col space, row space.

· Proving a subset is / is not a subspace.

strategy: $\begin{cases} \text{"is not": look for a counter-example showing the subset violates some subspace property.} \\ \text{"is": check all the subspace properties or realize the given set as a span / kernel ...} \end{cases}$

Today: · null space of a matrix.

· basis of a subspace.

· properties of basis, computation of basis.

1. Null space

Let A be an $m \times n$ matrix.

Def. We define the null space of A to be the set $\{ x \in \mathbb{R}^n \mid Ax = 0 \} \subseteq \mathbb{R}^n$.

We denote the null space by $\mathrm{Null}(A)$.

Prop: $\mathrm{Null}(A)$ is a subspace of $\mathbb{R}^n$.

pf: **Method 1.** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be the associated linear map with $T(x) = Ax$.

Then $\mathrm{Null}(A) = \{ x \in \mathbb{R}^n \mid T(x) = 0 \} = \ker T$.

Since kernels of linear maps are subspaces of their domains,

$\mathrm{Null}(A) = \ker(T)$ is a subspace of $\mathbb{R}^n$.

Method 2. Check the subspace axioms. E.x.

(a) $0 \in \mathrm{Null}(A)$?

(b) $u, v \in \mathrm{Null}(A) \overset{?}{\Longrightarrow} u+v \in \mathrm{Null}(A)$

(c) $u \in \mathrm{Null}(A)$, $c \in \mathbb{R}$

$\Downarrow ?$

$cu \in \mathrm{Null}(A)$

2. Basis

The examples show that a subspace V often have many diff bases.

Def: Let V be a subspace of $\mathbb{R}^n$ (possibly $V = \mathbb{R}^n$). A basis of V is a subset of V that (i) is linearly independent and (ii) spans V.

Examples/Nonexamples: (1) Take $n = 2$, $V = \mathbb{R}^2$. The set $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is a basis (e_1, e_2):

(i). lin ind: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, all cols are pivot ✓ (or: $a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} = 0 \Rightarrow a = b = 0$)

(ii) span: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ EF, no zero row ✓ (or. $\begin{bmatrix} a \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix}$)

Or, use the invertible matrix theorem, check one of (i) and (ii), and note that the theorem implies that the other must also hold.

(2) $V = \mathbb{R}^2$, $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$ $\quad \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & -4 \end{bmatrix}$ EF, no zero row.

It follows that $\mathcal{B}$ spans $\mathbb{R}^2$, which further implies $\mathcal{B}$ is lin ind by the theorem. So $\mathcal{B}$ is a basis.

(3). $V = \mathbb{R}^2$. $B' = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix} \right\}$. $B'' = \left\{ \begin{bmatrix} 1 \\ \pi \end{bmatrix} \right\}$

Recall that a set of 3 vectors in $\mathbb{R}^2$ cannot be lin. ind. since $3 > 2$, so B' is not a basis. Also, a set of 1 vector in $\mathbb{R}^2$ cannot span $\mathbb{R}^2$ since $1 < 2$, so B'' cannot be a basis of $\mathbb{R}^2$.

Same logic.

Prop: A basis B in $\mathbb{R}^n$ must have exactly n distinct vectors.

Furthermore, if B has n vectors $\{v_1, v_2, \ldots, v_n\}$ (eg. $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ in $\mathbb{R}^2$), then B is a basis of $\mathbb{R}^n$ iff the matrix $C = [v_1 | \cdots | v_n]$ is invertible, iff $EF(C)$ has no zero rows, iff ......

(4). $V = \underbrace{\text{Span}}_{\text{subspace}} \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\} \subseteq \mathbb{R}^n$.

Is $B = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$ a basis of V? Yes: (i) Being a set containing a single nonzero vector, B must be lin. ind. (ii) B certainly spans V.

Is $B' = \left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\}$ a basis of V? No: (i). B' is still lin. ind, but (ii) $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ doesn't span $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$: everything in $\text{Span}\, B' = \text{Span}\left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\}$ is of the form $\begin{bmatrix} 2c \\ 0 \end{bmatrix}$, which has zero as the 2nd coordinate, so $\text{Span}\, B' \neq V$.

Is $B'' = \left\{ \begin{bmatrix} 6 \\ 2 \end{bmatrix} \right\}$ a basis of V? Yes: (i) B'' is lin ind as before.
(ii). Take $v \in V$, then $v = c\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \frac{1}{2}c \begin{bmatrix} 6 \\ 2 \end{bmatrix} \in \text{Span}\, B''$. so V is spanned by B''.

2.2. A uniqueness theorem.

<u>Thm</u>: Let V be a subspace of $\mathbb{R}^n$ and let $\mathcal{B} = \{v_1, \cdots, v_k\}$ be a basis of V. Then for all $v \in V$, there are unique scalars $c_1, \cdots, c_k$ s.t.

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k \qquad (*)$$

<u>Example</u>: $V = \mathbb{R}^2 \subseteq \mathbb{R}^2$. $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ $\quad v = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$. $\quad v = \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \underset{\text{has to be } 5.}{\overset{\text{has to be } 3}{?}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + ? \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

has to be 3

has to be 5.

$\mathcal{B}' = \left\{ \underset{v_1}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} 1 \\ -2 \end{bmatrix}} \right\}$. $\quad v = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$, <u>Q</u>: How can we find x, y s.t.

$$v = x v_1 + y v_2, \text{ i.e., } x \begin{bmatrix} 1 \\ 2 \end{bmatrix} + y \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \text{ i.e. } \underbrace{\begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}}_{C} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}.$$

C, inv. since $\mathcal{B}$ is a basis

unique sol $\begin{bmatrix} x \\ y \end{bmatrix} = C^{-1} \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ ...... $x = \frac{3}{2}$, $y = \frac{3}{2}$.

Pf: Method 1. Use the idea illustrated by the example, talk about matrix equations and the inv. matrix theorem ... ...

Method 2. (I). A decomp. in the form (*) must exist since $\mathcal{B}$ spans V and $v \in V$.

(existence)

(II) Say $v \overset{ⓐ}{=} c_1 v_1 + \cdots + c_k v_k$ and also $v \overset{ⓑ}{=} d_1 v_1 + \cdots + d_k v_k$

(uniqueness) we need to show that $c_1 = d_1$, $c_2 = d_2$, ..., $c_k = d_k$.

ⓐ − ⓑ $\quad 0 = v - v = (c_1 - d_1) v_1 + \cdots + (c_k - d_k) v_k$

Since $\mathcal{B}$ is lin ind, this implies that $c_1 - d_1 = c_2 - d_2 = \cdots = c_k - d_k = 0$,

i.e., $c_1 = d_1$, $c_2 = d_2$, ..., $c_k = d_k$. □

2.3. Computing bases of kernels / null spaces.

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map and let A be the standard matrix of T, so that $T(x) = Ax$ $\forall x \in \mathbb{R}^n$ and $\ker T = \mathrm{Null}(A) = \{ x \in \mathbb{R}^n \mid Ax = 0 \}$

Recall that $\mathrm{Null}(A)$ has a p.v.f. (e.g. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x - y \to \mathrm{Null}(A) = \left\{ y \begin{bmatrix} 1 \\ 1 \end{bmatrix} : y \in \mathbb{R} \right\}$.

Thm: The constant vectors in the p.v.f. always forms a basis of $\mathrm{Null}(A)$.

Next time:
- example computations
- bases of other kinds of subspaces.