Math 2130. Lecture 20.

03.05.2021.

Last time:

- Matrix inversion algorithm (also determines invertibility)
- Inverse of linear maps
- Subspaces of $\mathbb{R}^n$: definition and first examples

Today: More examples and nonexamples, (writing) proofs.

Recall that a subset S of $\mathbb{R}^n$ is called a <u>subspace</u> of $\mathbb{R}^n$ if

(a). $0 \in S$,

(b) $v, w \in S \Rightarrow v + w \in S$,

and (c) $v \in S$, $c \in \mathbb{R} \Rightarrow cv \in S$.

Examples / Non-examples, with proofs

Let n be an integer.

(a). The zero/trivial subspace: $S = \{0\} \in \mathbb{R}^n$

We saw that $\{0\}$ is always a finite subspace of $\mathbb{R}^n$.

We call it the zero/trivial subspace. (an "extreme" subspace: the smallest possible)

(b). If a subspace $S \subseteq \mathbb{R}^n$ contains any nonzero vector v, then the inf. many multiples of v must all be in S.

e.g. $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \in S \subseteq \mathbb{R}^2$.

$c \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} c \\ 2c \end{bmatrix}$ should be in S $\forall c \in \mathbb{R}$.

Thus, any nontrivial subspace must be infinite.

In other words, the only finite subspace of $\mathbb{R}^n$ is $\{0\}$.

(c) $S = \mathbb{R}^n \subseteq \mathbb{R}^n$. The entire subset $\mathbb{R}^n$ is certainly a subspace of $\mathbb{R}^n$ (the other extreme subspace: the largest possible)

(a) ✓ (b) ✓ (c) ✓

(d). Spans. We saw that for any one vector $v \in \mathbb{R}^n$, the span of v $\mathrm{Span}\{v\} = \{cv \mid c \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^n$.

In fact, for any finite subset $A = \{v_1, \dots, v_k\}$, the span of A

$$\mathrm{Span}(A) = \mathrm{Span}\{v_1, \dots, v_k\} = \{c_1 v_1 + \dots + c_k v_k \mid c_1, \dots, c_k \in \mathbb{R}\}$$

is always a subspace of $\mathbb{R}^n$. e.g. $n=3$, $k=2$, $A = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

Pf: We check that $\mathrm{Span}(A)$ satisfies the conditions (a) — (c).

(a) ($0 \in \mathrm{Span}\, A$? Is 0 a lin. comb of $v_1, \dots, v_k$?)

$$0 = 0 \cdot v_1 + 0 \cdot v_2 + \dots + 0 \cdot v_k \in \mathrm{Span}\, A.$$ So A is satisfied.

b) (if $v, w \in \mathrm{Span}(A)$, is it true that $v+w \in \mathrm{Span}(A)$?)

Suppose $v, w \in \mathrm{Span}(A)$, then $v = c_1 v_1 + \cdots + c_k v_k$, $w = d_1 v_1 + \cdots + d_k v_k$ for some $c_1, \ldots, c_k, d_1, \ldots, d_k \in \mathbb{R}$. But then

$$v+w = (c_1 v_1 + \cdots + c_k v_k) + (d_1 v_1 + \cdots + d_k v_k) = (c_1+d_1)v_1 + \cdots + (c_k+d_k)v_k .$$

So $v+w \in \mathrm{Span}(A)$, hence condition (b) is satisfied.

c). Ex.

Def: Let $A = \begin{bmatrix} r_1 \\ \vdots \\ r_m \end{bmatrix} = [c_1 | \cdots | c_n]$ be a $m \times n$ matrix. The <u>column space</u> of A is the span of its cols, and the <u>row space</u> of A is the span of $\underbrace{\text{it rows}}_{r_i^T}$.

Note: Our discussion above implies that the col. space $\mathrm{Col}(A)$ is a subspace of $\mathbb{R}^m$ and the row space $\mathrm{Row}(A)$ is a subspace of $\mathbb{R}^n$.

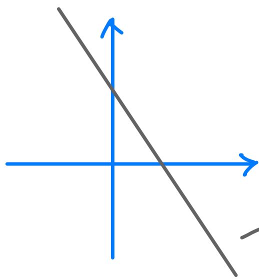
e.g. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

$\nearrow$ $\operatorname{Col} A = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix} \right\} \subseteq \mathbb{R}^2$

$\searrow$ $\operatorname{Row} A = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\} \subseteq \mathbb{R}^3$.

e). Some geometric (non) examples.

(i)

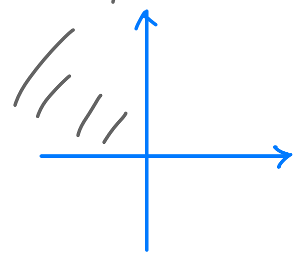


S: a line in $\mathbb{R}^2$ not through the origin

Note: S is not a subspace of $\mathbb{R}^2$ since $0 \notin S$.

(ii)

$S' = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : \begin{matrix} x \leq 0 \\ y \geq 0 \end{matrix} \right\}$



(a). ✓

(b). ✓

(c). ✗.

Note that $\begin{bmatrix} -1 \\ 1 \end{bmatrix} \in S'$ but $(-1)\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \notin S'$, so S' is not a subspace.

(f). Soln sets of matrix equations (*) $\underset{m\times n}{A}x = b$. (Running example: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$)

Two cases: (1) $b = 0$, i.e., (*) is homogeneous, then we claim that the soln set S of (*) is a subspace. Pf: (a). $A \cdot 0 = 0 = b$, so $0 \in S$.

(b). if $x, y \in S$, then $Ax = 0$, $Ay = 0$, so $A(x+y) = Ax + Ay = 0 + 0 = 0$, so $x + y \in S$.

(c). if $x \in S$ and $c \in \mathbb{R}$, then $A(cx) = cAx = c \cdot 0 = 0$. so $cx \in S$.

Having checked (a) – (c), we conclude that S is a subspace of the $\mathbb{R}^n$.

(2) $b \neq 0$, i.e., (*) is non-homogeneous, then $A \cdot 0 = 0 \neq b$ so 0 is not in the soln set S of (*), so S is not a subspace of $\mathbb{R}^n$.

Conclusion: The soln set S of (*) is a subspace of $\mathbb{R}^n$ iff $b = 0$.

(g). Kernel of linear maps: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear map.

Prop: $\operatorname{Ker} T$ is a subspace of $\mathbb{R}^n$.

Pf: Method 1. Check (a) – (c). E.x.

necessarily $m \times n$.

Method 2. Consider the standard matrix A of T, so that $T(x) = Ax$,

So $\operatorname{Ker} T = \{ x \in \mathbb{R}^n \mid T(x) = 0 \} = \{ x \in \mathbb{R}^n \mid Ax = 0 \}$ = Soln set of the hom. eq $A\vec{x} = \vec{0}$,

so by the previous example, $\operatorname{Ker} T$ is a subspace of $\mathbb{R}^n$.

(h). Image of linear maps: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear map.

Prop: $\operatorname{Im} T$ is a subspace of $\mathbb{R}^m$.

Method 1. Check (a) – (c). E.x.

standard matrix

Method 2. Show that $\operatorname{Im} T = \operatorname{Col}(A)$, then recall that col spaces form subspaces of suitable spaces.

Summary: The following sets are all subspaces of $\mathbb{R}^n$.

- 0 and $\mathbb{R}^n$
- the span of any subsets of $\mathbb{R}^n$. ($A \to \operatorname{Span} A$) typically infinite, even if A is finite
- soln sets of homogeneous eq. $Ax = 0$ where A has n cols.
- kernels of linear maps from $\mathbb{R}^n$ to some $\mathbb{R}^m$,

  and images of linear maps from some $\mathbb{R}^p$ to $\mathbb{R}^n$.