Math 2130. Lecture 2

01/20/2021

Last time:

- Matrix encoding of systems of linear equations (L.E.S.)
- New scheme for solving L.E.S.

$$
\begin{tikzcd}
\text{given L.E.S.} \arrow[r, "\text{encode}"] \arrow[d, "\text{old: method of elimination}"'] \arrow[dr, phantom, "\text{new}"] & \text{a matrix } A \arrow[d, "\text{row operations}"] \\
\text{the soln(s)} & \text{a ``nicer'' matrix } A' \arrow[l]
\end{tikzcd}
$$

"nicer": encodes an equiv. easier to get the solns from the matrix

- The row ops. are motivated by the method of elimination.
  They are ...... (next page)

1. Elementary row operations.

Def: By an <u>elementary row operation</u> on a matrix A, we mean one of the following three kinds of operations:

(E1) (Interchange) Swap two rows. $R_i \leftrightarrow R_j$

(E2) (Scaling) Multiply (every entry in) a row by a nonzero constant c.
$R_i \to cR_i$

(E3) (Replacement) Add a multiple of a row R_j to another row R_i to replace R_i. $R_i \leftarrow R_i + c \cdot R_j$

Prop: (Elementary row ops preserve equivalence of L.E.S.s)

If a seq. of elt row op. take a matrix A to another A', then A and A' encode equivalent L.E.S.'s.

Unanswered questions:

1. What counts a "nicer" matrix A' from Page 1?

A: "Echelon form" $\longrightarrow$ matrices in echelon form make solns easy to find.

2. Can we always get to the nicer encoding matrix via only the three elt. row operations? If so, how?

A: Yes, we can always tranform a matrix A to an echelon form by an algorithm called Gaussian elimination.

2. Another example (from the last lecture)

(1). $\begin{cases} 3z = 9 \\ 2x - z = 5 \\ 2y + z = 1 \end{cases} \longrightarrow \begin{bmatrix} 0 & 0 & 3 & 9 \\ 2 & 0 & -1 & 5 \\ 0 & 2 & 1 & 1 \end{bmatrix} \xrightarrow{E1} \begin{bmatrix} 2 & 0 & -1 & 5 \\ 0 & 0 & 3 & 9 \\ 0 & 2 & 1 & 1 \end{bmatrix}$

$$\xrightarrow{E1} \begin{bmatrix} 2 & 0 & -1 & 5 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 3 & 9 \end{bmatrix} \xrightarrow{E2} \begin{bmatrix} 2 & 0 & -1 & 5 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow[R2 \leftarrow R2 - R3]{E3} \begin{bmatrix} 2 & 0 & -1 & 5 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

this turns out to be in "echelon form".

$$\xrightarrow[R_1 \leftarrow R_1 + R_3]{E3} \begin{bmatrix} 2 & 0 & 0 & 8 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow{E2} \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \longrightarrow \begin{cases} x = 4 \\ y = -1 \\ z = 3 \end{cases}$$

3. Echelon forms.

Def. A matrix is in <u>echelon form</u> if it satisfies:

(1) All nonzero rows appear above any <u>zero row</u>.
(all entries are zero)

(2) Every <u>leading entry</u> in a row is in a column strictly to the
(= first nonzero entry)
right of the column of the leading entry in the above row.

(3) All entries in a column below a row leading entry are zeros.

(In fact, (2) and (3) are equivalent.)

If a matrix A is in echelon form, then we say A is in <u>reduced echelon form</u> if it further satisfies the conditions.

(a). Every row-leading entry is 1.

(b). Every entry above a row-leading entry in a column is zero. (so that row-leading entry is the only nonzero entry in the column, because entries below it are zero by (3).)

2. Another example (from the last lecture)

↙ copied page.

(1). $\begin{cases} 3z = 9 \\ 2x - z = 5 \\ 2y + z = 1 \end{cases} \longrightarrow \begin{bmatrix} 0 & 0 & 3 & 9 \\ 2 & 0 & -1 & 5 \\ 0 & 2 & 1 & 1 \end{bmatrix} \xrightarrow{E1} \begin{bmatrix} 2 & 0 & -1 & 5 \\ 0 & 0 & 3 & 9 \\ 0 & 2 & 1 & 1 \end{bmatrix}$

$$\xrightarrow{E1} \begin{bmatrix} 2 & 0 & -1 & 5 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 3 & 9 \end{bmatrix} \xrightarrow{E2} \begin{bmatrix} 2 & 0 & -1 & 5 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow[R2 \leftarrow R2 - R3]{E3} \begin{bmatrix} 2 & 0 & -1 & 5 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

✓ echelon form. not reduced

this turns out to be in "echelon form". echelon!

reduced echelon form ↙

$$\xrightarrow[R_1 \leftarrow R_1 + R_3]{E3} \begin{bmatrix} 2 & 0 & 0 & 8 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow{E2} \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \longrightarrow \begin{cases} x = 4 \\ y = -1 \\ z = 3 \end{cases}$$

Examples / Non-examples.

Below we use $\blacksquare$ to stand for nonzero numbers and $*$ for arbitrary numbers.

(1). $\begin{bmatrix} 0 & 0 & 0 \\ \blacksquare & * & * \end{bmatrix}$ not echelon, zero row above nonzero row.

$\downarrow$ E1

(2). $\begin{bmatrix} \blacksquare & * & * \\ 0 & 0 & 0 \end{bmatrix}$ echelon, not reduced echelon unless $\blacksquare = 1$

(3). $\begin{bmatrix} \blacksquare & * & * & \pi & 0 & 3 \\ 0 & \blacksquare & * & * & 0 & 5 \\ 0 & 0 & 0 & 0 & \blacksquare & 0 \end{bmatrix}$ echelon, reduced echelon if and only if all the three $\blacksquare$'s equal 1 and $* = 0$.

Theorem.

(1) Every matrix can be transformed to a matrix in echelon form.

Moreover, we can do so using only the three elementary row operations

(2) Every matrix can be transformed to a unique (one and only one) matrix in reduced echelon form.

Reason: Property (1) can be acheived using E1,

... (2)/(3) E3,

(a) E2,

(b) .... E3.

(the uniqueness of the reduced echelon form is harder.)

Ex. Solve $\begin{cases} x+y+z=6 \\ x+2y+z=8 \\ 2y-z=-2 \end{cases}$ by transforming its encoding matrix into its reduced echelon form using elt. row ops.

Next time: Row reduction algorithm to get R.E.F.