Math 2130. Lecture 19.

03.03.2021.

Last time: · Algorithm for checking invertibility and computing inverses

Today: · more example applications, justifying the algorithm.

· Invertible linear maps. Relation to matrix inversion.

· Start §2.8. Subspaces of $\mathbb{R}^n$.

1. Review of the algorithm

E.g. $A = \begin{bmatrix} 2 & 1 \\ 0 & -3 \end{bmatrix} \rightsquigarrow B = \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & -\frac{1}{3} \end{array}\right] \xrightarrow{\text{go on}} \left[\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & -\frac{1}{3} \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & \frac{1}{6} \\ 0 & 1 & 0 & -\frac{1}{3} \end{array}\right]$

EF, no zero row $\Rightarrow$ A is inv

$I_2 \Longrightarrow A^{-1}$

finishes §2.3.

So $A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{6} \\ 0 & -\frac{1}{3} \end{bmatrix}$.

Why the algorithm works:

(1) Every elt row op can be realized by left mult by a matrix.

(a). Scaling by a nonzero number c

$$\underbrace{\begin{bmatrix} 2 & 1 \\ 0 & -3 \end{bmatrix}}_{A} \longrightarrow \underbrace{\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}}_{E} = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{3} \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 0 & -3 \end{bmatrix}$$

(b). interchanging two rows. e.g.

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}}_{A} \longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 4 & 5 & 6 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{E} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

(c). adding a multiple of a row to another row.

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{A} \begin{matrix} \times(-3) \\ \downarrow + \end{matrix} \longrightarrow \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}}_{E} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Fact: For each elt row op, we can find a matrix E such that apply the op to A results in EA.

2). Matrices of the form E mentioned in 1) are all invertible.

$\underbrace{\text{Matrices of the form } E \text{ mentioned}}_{\text{"elementary row matrices"}}$

3). Why the algorithm works:

if $[A \mid I] \xrightarrow{\text{row ops}} [I \mid C]$, then we can find elt row matrices $E_1, E_2, \cdots, E_r$ s.t.

$$\begin{cases} E_r \cdots E_2 E_1 A = I & ① \\ E_r \cdots E_2 E_1 I = C & ② \end{cases}$$

Now, $① \Rightarrow A = (E_r \cdots E_1)^{-1} = E_1^{-1} \cdots E_r^{-1}$

So A is invertible and $A^{-1} = (E_1^{-1} \cdots E_r^{-1})^{-1} = E_r E_{r-1} \cdots E_2 E_1 \overset{②}{=} C.$

□

2. Invertible linear maps.

Def: A linear map $T: \mathbb{R}^n \to \mathbb{R}^n$ is called <u>invertible</u> if there is a linear map $S: \mathbb{R}^n \to \mathbb{R}^n$ s.t. $S \circ T = \underline{Id_{\mathbb{R}^n}}$ and $T \circ S = Id_{\mathbb{R}^n}$

where $Id_{\mathbb{R}^n}$ denotes the identity map with $Id_{\mathbb{R}^n}(x) = x \quad \forall x \in \mathbb{R}^n$.

Thm: Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear map and let A be its standard matrix. (e.g. $T: \mathbb{R}^2 \to \mathbb{R}^2$, $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$). Then the following are equivalent: (1) T is inv. (2). A is inv.

(3) T is surj (4) T is inj (5) T is both inj and surj.

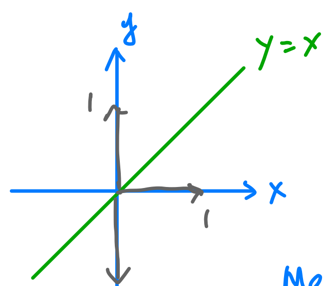
(6) $EF(A)$ has zero rows. ... .

Moreover, if T is invertible (hence A is inv.), then the standard matrix of T^{-1} is A^{-1}, i.e., $T^{-1}(x) = A^{-1}\cdot x$. $\square$

Eg. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the map that first reflects points across $y = x$ and then reflects points across the x-axis. Is T invertible? If so, find the formula for the inverse map.

$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as usual

Sol:



We first find the standard matrix $A = \big[\, T(e_1) \,\big|\, T(e_2) \,\big]$ of T.

$$T(e_1) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad T(e_2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \text{so } A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Method 1. $0\cdot 0 - 1\cdot(-1) = 1 \neq 0$, so A is inv and $A^{-1} = \frac{1}{1}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Method 2. $$B = [A \mid I] = \left[\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{array}\right] \to \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \to \left[\begin{array}{cc|cc} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{array}\right],$$

I_2

so A is inv and $A^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

By the theorem, $T^{-1}(v) = A^{-1} \cdot v$, i.e., $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$.

$\square$

E.g. Find the inverse map of the linear map $T: \mathbb{R}^2 \to \mathbb{R}^2$, $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x-3y \\ y \end{bmatrix}$.

__Soln:__ Note that the standard matrix A of T is $\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$.

(two ways to get A. $\begin{cases} \text{① note that } \begin{bmatrix} x-3y \\ y \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ \text{② compute } A = [\, T(e_1) \mid T(e_2) \,] = \left[T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \middle| T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \right] = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}. \end{cases}$)

E.x. find A^{-1}, then compute $T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A^{-1} \begin{bmatrix} x \\ y \end{bmatrix}$.

3. Subspaces of $\mathbb{R}^n$

The set $\mathbb{R}^n$ is an example of a "vector space".

We'll define subspaces of $\mathbb{R}^n$.

Def: A subset $S \subseteq \mathbb{R}^n$ of $\mathbb{R}^n$ is called a <u>subspace</u> of $\mathbb{R}^n$

if (1) the zero vector $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ of $\mathbb{R}^n$ is in S.

(2) for all $u, v \in S$, we have $u+v \in S$ as well. "closure under addition"

and (3) for all $v \in S$ and $c \in \mathbb{R}$, we have $cv \in S$ as well. "closure under scaling"

Examples/Nonexamples. (a). $n=2$. $S = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$: S is not a subspace;

it violates all of (1) – (3)

1b) $S = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \subseteq \mathbb{R}^2$: not a subspace of $\mathbb{R}^2$; it satisfies condition (1) but fails conditions (2) and (3).

(2) take $u = v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then $u+v = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \notin S$.

(3) take $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $c = 3$, then $cv = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \notin S$.

Next time:

More basic proofs about subspaces.

1c). $S = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} \subseteq \mathbb{R}^2$: S satisfies all of (1) − (3), so it's a subspace.

1d) For any n and any $v \in \mathbb{R}^n$, the set $S = \mathrm{Span}\{v\} = \{cv \mid c \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^n$: (1). $0 = \underset{c}{0} \cdot v \in S$ ✓ (2) $c_1 v + c_2 v = \underbrace{(c_1 + c_2)}_{c} v \in S$. ✓

(3) $d(cv) = \underline{(dc)}\, v \in S$. ✓