Math 2130. Lecture 18.

03. 01. 2021.

Last time: Theoretical aspects of invertibility:

The invertible matrix theorem, including echelon form criteria for invertibility and relations between invertibility and lin ind., spanning sets, inj. maps. surj. maps, etc.

Today:

- Computational aspects:

  Algorithm for testing invertibility & finding inverses.

- Invertible linear maps. Relation to matrix inversion.

1. Finding matrix inverses.

Let A be an $n \times n$ matrix. e.g. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Here's an algorithm for both testing invertibility and finding inverses. ↙

Step 1: Form the matrix $B = [A \mid I_n]$, e.g. $\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right]$

Step 2: Use elt. row operations to transform B until the left ("A") half is in echelon form. (so the row ops should be motivated by the A-part, but the operations should also be done on the right part)

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \begin{matrix} \times(-3) \\ \downarrow + \end{matrix} \longrightarrow \left[\begin{array}{c|cc} \begin{matrix} 1 & 2 \end{matrix} & 1 & 0 \\ \underbrace{\begin{matrix} 0 & -2 \end{matrix}} & -3 & 1 \end{array}\right]$$

Step 3: If the resulting left part EF(A) EF(A) from step 2 has a zero row, then A is not invertible (so we are done). If EF(A) has no zero row, then A is invertible, and we continue to step 4.

Step 4. $[A \mid I] \to [\underbrace{EF(A)}_{\text{no zero row}} \mid C]$. To find A^{-1}, continue reducing the matrix $[EF(A) \mid C]$ from Step 2 to the point where the left (A) part is in reduced echelon form $REF(A) \overset{\text{necessarily}}{=} I_n$.

The right half in the final result will be A^{-1}.

$$\left[\begin{matrix}1&2\\3&4\end{matrix}\;\middle|\;\begin{matrix}1&0\\0&1\end{matrix}\right] \to \left[\underbrace{\begin{matrix}1&2\\0&-2\end{matrix}}_{EF(A)}\;\middle|\;\underbrace{\begin{matrix}1&0\\-3&1\end{matrix}}_{C}\right] \to \left[\begin{matrix}1&2\\0&1\end{matrix}\;\middle|\;\begin{matrix}1&0\\\frac{3}{2}&-\frac{1}{2}\end{matrix}\right] \to \left[\underbrace{\begin{matrix}1&0\\0&1\end{matrix}}_{I_2}\;\middle|\;\underbrace{\begin{matrix}-2&1\\\frac{3}{2}&-\frac{1}{2}\end{matrix}}_{\substack{\downarrow\\ \text{must be}\\ A^{-1}.}}\right]$$

So $A^{-1} = \begin{bmatrix}-2&1\\\frac{3}{2}&-\frac{1}{2}\end{bmatrix}$. ✓

E.g. $A = \begin{bmatrix} 0 & 3 & -5 \\ 1 & 0 & 2 \\ -4 & -9 & 7 \end{bmatrix}$. Determine if A is inv, and find A^{-1} if so.

Sol: $$B = \left[\begin{array}{ccc|ccc} 0 & 3 & -5 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ -4 & -9 & 7 & 0 & 0 & 1 \end{array}\right] \updownarrow \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 3 & -5 & 1 & 0 & 0 \\ -4 & -9 & 7 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \times 4 \\ \downarrow + \\ \end{array}$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} \textcircled{1} & 0 & 2 & 0 & 1 & 0 \\ 0 & 3 & -5 & 1 & 0 & 0 \\ 0 & -9 & 15 & 0 & 4 & 1 \end{array}\right] \begin{array}{l} \times 3 \\ \downarrow + \end{array} \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 3 & -5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 4 & 1 \end{array}\right].$$

The left half is in echelon form but has a zero row., ie, EF(A) has a zero row, so by the inv. matrix thm A is not invertible. □

Next time: more inverse computations. why the alg. works.