Math 2130. Lecture 17.

02/26/2021.

Last time:

- invertibility vs. cancellation: $\left.\begin{array}{l} Ax = b \\ A \text{ inv.} \end{array}\right\} \to x = A^{-1}b$
- properties of matrix inverses: $(A^{-1})^{-1} = A$, $(AB)^{-1} = B^{-1}A^{-1}$, $(A^T)^{-1} = (A^{-1})^T$.
- characterization of invertible matrices:

Important construction: A, $n\times n$ matrix $\rightsquigarrow$ $T_A : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto Ax$ (linear map)

(so A is the standard matrix of T_A).

A is inv $\Longrightarrow$ T_A is injective

$\Updownarrow$

the cols of A are lin ind.

In fact, "$\Leftarrow$" also works by the "Invertibility Thm" . . .

The Inv. Thm. **(§ 2.3. Thm 8)** Let A be an $n\times n$ matrix. The following are equivalent.

(a) A is invertible.

(b) The cols of A are lin ind.

(c) The cols of A span $\mathbb{R}^n$.

(d) All cols of A are pivot, i.e. $EF(A)$ has a pivot in every col.

(e) All rows of A are pivot, i.e., $EF(A)$ has no zero row.

(f) The map T_A is inj.

(g) The map T_A is surj.

(h) The map T_A has $\ker T_A = \{0\}$.

(i) The map T_A has $\operatorname{Im} T_A = \mathbb{R}^n$.

Rmks: (1) We already know by Ch. 1 that (b) $\Leftrightarrow$ (d) $\Leftrightarrow$ (f) $\Leftrightarrow$ (h) and (c) $\Leftrightarrow$ (e) $\Leftrightarrow$ g $\Leftrightarrow$ (i).

(2). In fact, we'll see that these nine conditions are equivalent to a tenth condition $\boxed{(a)' : \quad REF(A) = I_n}$.

Question: Why are the two sides of | equivalent?

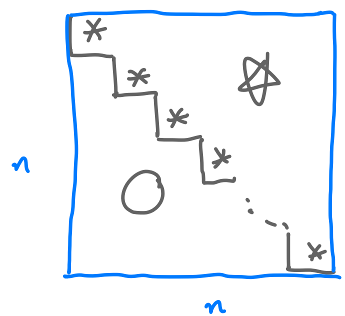
Answer: Because A is square:

(c) holds $\Longleftrightarrow$ # pivots of A = # cols of A = n

$\Longleftrightarrow$ # pivots of A = n = # rows of A

$\Longleftrightarrow$ every row in A has a pivot, i.e. (d) holds.

In picture: (c) $\rightarrow$



$\rightarrow$ every row has a pivot, i.e. (d) holds

echelon form criterion!

($*$: nonzero number)

$\Longleftrightarrow$ (a'): $REF(A) = I_n$.

So, we've argued that (a') $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) $\cdots$ $\Leftrightarrow$ (i).

e.g.

$$A = \begin{bmatrix} 1 & 7 \\ -3 & -6 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & 7 \\ 0 & 15 \end{bmatrix}}_{EF(A)}$$

$$\begin{array}{cc} \downarrow & \downarrow \\ p & p \end{array}$$

Since there's a pivot in every col of $EF(A)$, A must be invertible.

$$A' = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}}_{EF(A')}.$$

Since there's a zero row in $EF(A)$, A cannot be invertible.

Rmk: We haven't explained why (a) is implied by the conditions (a'), (b) – (i).

We haven't explained how to find A^{-1} for an inv. matrix A either. We'll do these things next week.