Math 2130. Lecture 16.

02. 24. 2021.

Last time:

- Powers of square matrices
- Matrix transposition
- Inverses of square matrices.

Today. More on inverses.

- Using invertibility for cancellation.
- Computation and properties of inverses.
- Characterization of invertibility.

1. Invertibility and cancellation.

Easy fact: $I_n \cdot v = v$ for any $v \in \mathbb{R}^n$

Thm 1. Let A be an $n \times n$ matrix. If A is invertible, then for every $b \in \mathbb{R}^n$, the matrix equation $Ax = b$ has a unique soln, namely $x = A^{-1} b$.

Pf: (1) $A^{-1}b$ is a soln: $A(A^{-1}b) = (AA^{-1})b = I_n b = b$. ✓

(2) $A^{-1}b$ is the only soln: $Ax = b \Rightarrow \underset{(A^{-1}A)x}{\underset{\parallel}{A^{-1}(Ax)}} = A^{-1}b \Rightarrow I_n x = A^{-1}b \Rightarrow x = A^{-1}b$ ✓. □

Eg. $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$, $B = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$. Recall that $B = A^{-1}$. Take $b = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$.

By Thm 1, the equation $Ax = b$, i.e., $\begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} x = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$ has a unique solution

$$x = A^{-1}b = Bb = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} -56 \\ 23 \end{bmatrix}.$$

(check: $\begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} \begin{bmatrix} -56 \\ 23 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$.)

Corollary of Thm 1. Consider the map $T_A : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto Ax$.

$(A: n\times n)$ (note that the standard matrix of T_A is A by design)

Now suppose that A is invertible. Then for every $b \in \mathbb{R}^n$, there is a unique x s.t. $Ax = T_A(x) = b$. So T_A is bijective. i.e.,

the linear map naturally associated to an inv. matrix is bijective.

e.g. $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$ is invertible, so the map $T_A : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x+5y \\ -3x-7y \end{bmatrix}$ is bijective.

In particular, since $Ax=0$ has a unique sol (necessarily $x=0$), the cols of A are lin ind. i.e.,

bijective = both inj and surj.

the cols of an invertible matrix are always linearly ind.

2. Computation and properties of inverses.

(a) Inverses of 2×2 matrices.

Thm 2. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix where $ad - bc \neq 0$. Then

A is invertible, with
$$A^{-1} = \underbrace{\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}_{B}.$$

Pf: We need to check that $AB = I_2$ and $BA = I_2$. $\rightarrow$ HW.

E.g. Let $A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$. Calculate A^{-1}, then solve $Ax = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$.

Sol:
$$A^{-1} = \frac{1}{18-20}\begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix} = \frac{1}{-2}\begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ \frac{5}{2} & -\frac{3}{2} \end{bmatrix}.$$

$$Ax = \begin{bmatrix} 3 \\ 7 \end{bmatrix} \Rightarrow x = A^{-1}\begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ \frac{5}{2} & -\frac{3}{2} \end{bmatrix}\begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}. \quad \square$$

Questions: (1) What if $ad-bc=0$?

Fact: If $ad-bc=0$, then $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is not invertible.

Challenge Problem: Can you explain why?

(2) What about larger $n \times n$ matrices? How can we tell if they are invertible? If so, how do we find their inverses?

A: We'll develop algorithms for solving these problems

b) Properties of inverses (relation to mult / transposition)

<u>Thm 3.</u> Suppose A, B are invertible $n\times n$ matrices. Then

(1) $(A^{-1})^{-1} = A$. $\leftarrow$ already saw this.

(2) AB is inv. Moreover, $(AB)^{-1} = B^{-1}A^{-1}$.

(similarly, BA is invertible, with $(BA)^{-1} = A^{-1}B^{-1}$.)

(3) A^T and B^T are both inv. with $(A^T)^{-1} = (A^{-1})^T$ and $(B^T)^{-1} = (B^{-1})^T$.

<u>Rmks:</u> · A word on proofs : (2). need $\left(\underbrace{B^{-1}A^{-1} \cdot AB}_{= B^{-1}I_nB = B^{-1}B}\right) \overset{\checkmark}{=} I_n$ and $AB \cdot (B^{-1}A^{-1}) \overset{\checkmark}{=} I_n$.

(3) need $\underbrace{(A^{-1})^T \cdot A^T}_{= (AA^{-1})^T = I_n^T} \overset{\checkmark}{=} I_n$ and $A^T \cdot (A^{-1})^T \overset{\checkmark}{=} I_n$.

"new inverse from old", similarly for (3).

· (2) says "if you have the inv. of A and B, you know the inverse of AB"

3. Characterization of invertibility

Easy fact: Not all square matrices are invertible ; on Page 3, we saw that an invertible matrix must have lin ind cols.

e.g. $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ $\to$ the cols are mult. of each other, hence dep so the matrix is not inv.

Q: Under what condition is a square matrix invertible?

Goal: Tie the notion of invertibility to notions we've studied, such as echelon forms, linear independence, surj of linear maps.

Thm 4. Let A be an $n \times n$ matrix. The following are equivalent.

(§ 2.3. Thm 8)

(a) A is invertible.

(b) The cols of A are lin ind.

(c) The cols of A span $\mathbb{R}^n$.

(d) All cols of A are pivot, i.e. $EF(A)$ has a pivot in every col.

(e) All rows of A are pivot, i.e., $EF(A)$ has no zero row.

(f) The map $T_A : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto Ax$ is inj.

(g) The map $T_A : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto Ax$ is surj.

(h) The map T_A from (f) (g) has trivial kernel, i.e. $\ker T_A = \{0\}$.

(i) The map T_A from (f), (g) has "full image", i.e. $\operatorname{Im} T_A = \mathbb{R}^n$.

Next time: we'll prove Thm 4 and discuss its applications