Math 2130. Lecture 15. Midterm I Review: online now! 02.22.2021.

Last time: Properties of matrix multiplication:

(1). (Whenever the following expressions make sense, we have)

- $A(B+C) = AB + AC \qquad (A+B)C = AC + BC$
- $(rA)B = r(AB) = A(rB)$
- $A(BC) = (AB)C$ easy to state, harder to prove $\rightarrow$ " $M_{S\circ T} = M_S \cdot M_T$ ".
- $I_m A = A = A I_n$ if A is $m \times n$.

(2). failures: in general, matrix mult is $\underbrace{\text{not commutative}}_{AB \neq BA}$ and has $\underbrace{\text{no cancellation law}}_{AB = AC \not\Rightarrow B = C,\ AB \neq 0 \not\Rightarrow (A=0 \text{ or } B=0)}$.

Today. · Two more operations: powers of matrices, transposes of matrices.

· Properties of transposition. · Intro to invertible matrices.

1. Matrix powers

Q: Given an $m\times n$ matrix A, under what conditions does $A\cdot A$ make sense?

A: The condition should be # cols of A = # rows of A, i.e., $m=n$, i.e., A is square.

Note: · When A is square, we may form the power $A^k = \underbrace{A\cdot A\cdot \cdots \cdot A}_{k \text{ copies}}$ for any int. $k\geq 0$.

· Just as we define $x^0 = 1$ for any nonzero number $x\in\mathbb{R}$, we define $A^0 = I_n$ for any $n\times n$ matrix A.

e.g. $A^4 = A\cdot A\cdot A\cdot A$

E.g. $A=\begin{bmatrix}1&2\\3&4\end{bmatrix}$. $\to$ $A^0 \overset{\text{def}}{=} \begin{bmatrix}1&0\\0&1\end{bmatrix}$ $A^1=A$ $A^2=\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}1&2\\3&4\end{bmatrix}=\begin{bmatrix}7&10\\15&22\end{bmatrix}$. $A^3=A^2\cdot A=\begin{bmatrix}37&\cdots\\\cdots&\cdots\end{bmatrix}$

2. Matrix transposition

Def: The <u>transpose</u> of an $m\times n$ matrix A is the $n\times m$ matrix B s.t. $B_{ij}=A_{ji}$, i.e., the matrix whose rows are the cols of A and whose cols are the rows of A.

We denote B by A^T.

Eg. $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow A^T=\begin{bmatrix} a & c \\ b & d \end{bmatrix}$.

$B=\begin{bmatrix} 1 & 2 & 1 \\ -1 & 3 & 5 \end{bmatrix} \longrightarrow B^T=\begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 1 & 5 \end{bmatrix}$

$(AB)^T=B^TA^T.$

$$\begin{cases} LHS=\left(\begin{bmatrix} a-b & 2a+3b & a+5b \\ c-d & 2c+3d & c+5d \end{bmatrix}\right)^T=\begin{bmatrix} a-b & c-d \\ 2a+3b & 2c+3d \\ a+5b & c+5d \end{bmatrix} \\ RHS=\begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 1 & 5 \end{bmatrix}\cdot\begin{bmatrix} a & c \\ b & d \end{bmatrix}=\begin{bmatrix} a-b & c-d \\ 2a+3b & 2c+3d \\ a+5b & c+5d \end{bmatrix} \end{cases}$$

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Prop. Assuming the following expressions make sense, we have

(a) $(A^T)^T = A$. , (b) $(A+B)^T = A^T + B^T$,

(c) $(rA)^T = rA^T \quad \forall r \in \mathbb{R}$, (d) $(AB)^T = B^T A^T$.

pf: (a) We know if A is $m\times n$, then A^T is $n \times m$, and hence $(A^T)^T$ is $m \times n$. so it suffices to show $[(A^T)^T]_{ij} = A_{ij}$ $\forall\, 1 \le i \le m,\ 1 \le j \le n$.

$$[(A^T)^T]_{ij} = [A^T]_{ji} = A_{ij}. \quad \checkmark$$

(b). (c). similar (but even easier)

$AB: m\times p \quad (AB)^T: p \times m$

(d). Say A is $m \times n$ and B is $n \times p$. Then $\forall\, 1 \le i \le p,\ 1 \le j \le m$

$$[(AB)^T]_{ij} = [AB]_{ji} = \mathrm{Row}_j(A)\cdot \mathrm{Col}_i(B) = \mathrm{Col}_j(A^T)\cdot \mathrm{Row}_i(B^T) = \mathrm{Row}_i(B^T)\cdot \mathrm{Col}_j(A^T) = [B^T A^T]_{ij}. \quad \square$$

3. Matrix Inverses.

Def: Let A be a square $n \times n$ matrix. We say that A is invertible if there is an $n \times n$ matrix B s.t. $\underbrace{AB = I_n = BA}_{*}$. In this case, we say that B is an inverse of A.

Point: if A is invertible, we can speak of the inverse of A and denote it by A^{-1}.

E.g. $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$, $B = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$.

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So $AB = I_2 = BA$, so A is invertible and B is ~~an~~ the inverse of A. $B = A^{-1}$.

Rmk: If A is invertible, then it must have a unique inverse: say B, B' are both inverses of A. then $B = B(\underbrace{AB'}_{I_n}) = BAB' = (\underbrace{BA}_{I_n})B' = B'$, so $B = B'$. $\square$

Remarks:

- Note that we only discuss invertibility for square matrices.
- If a square matrix A is invertible, say with inverse B. then $AB = I_n = BA$ for some n. Then $BA = I_n = AB$, so B is also invertible and A is its inverse. In other words, we have $(A^{-1})^{-1} = A$ if A is invertible.
- The notion of invertibility has to do with "cancelability" and is actually familiar: for numbers $2x = 6 \longrightarrow 1 \cdot x = 3$

$\left(\frac{1}{2} \cdot 2\right) x = \frac{1}{2} \cdot 6$ used 2 is invertible in that $\frac{1}{2} \cdot 2 = 1$

similarly.

if A is inv. then the matrix equation $Ax = b$ has a unique soln $x = A^{-1}b$

more next time.