Math 2130. Lecture 14.

Last time: · matrix operations: addition, scalar mult, multiplication.

Today: · properties of matrix multiplication

Warm-up: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 3 \\ 5 & 7 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 5 & 0 \\ 7 & -3 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Compute $A(-B)$, $-(AB)$, $A(B+C)$, $AB+AC$.

$$A(-B) = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 3 \\ 5 & 7 & 1 \end{bmatrix} \begin{bmatrix} -2 & -1 \\ -5 & 0 \\ -7 & 3 \end{bmatrix} = \begin{bmatrix} -33 & 8 \\ -23 & 8 \\ -52 & -2 \end{bmatrix}$$

$$-(AB) = -\begin{bmatrix} 33 & -8 \\ 23 & -8 \\ 52 & 2 \end{bmatrix} = \begin{bmatrix} -33 & +8 \\ -23 & +8 \\ -52 & -2 \end{bmatrix}$$

Note: $A(-B) = -(AB)$.

Similarly, $A(B+C) = AB+AC = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 3 \\ 5 & 7 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 6 & 0 \\ 7 & -2 \end{bmatrix} = \begin{bmatrix} 35 & \cdots \\ \cdots & \cdots \\ \cdots & \cdots \end{bmatrix}$.

1. Properties of mat mult.

Prop. (a) (Matrix mult interacts well with addition and scalar multiplication)

Given matrices A, B, C and scalar $c \in \mathbb{R}$, we have

$$A(B+C) = AB + AC, \quad (cA)B = c(AB) = A(cB)$$
$$(A+B)C = AC + BC,$$

whenever the products make sense.

(b) (Mat. mult not commutative in general!)

It is not generally true that $AB = BA$, even when both AB, BA are defined.

e.g. $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ $\rightarrow$ AB is defined, BA is not

$C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ $\rightarrow$ $DC = \begin{bmatrix} 2 & 4 \\ 9 & 12 \end{bmatrix}$, $CD = \begin{bmatrix} 2 & 6 \\ 6 & 12 \end{bmatrix} \neq DC$.

(c). (Matrix mult. is not defined coordinatewise even when possible).

e.g. $\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$

$= \begin{bmatrix} 2 & 2 \\ 5 & 5 \end{bmatrix}$.

(id). $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \end{bmatrix}$

$I_2 \cdot A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \end{bmatrix} = A$

similarlyly, $A = A \cdot I_3$.

(d). (There's no cancellation law for matrix multiplcation in general.)

$AB = 0 \not\Rightarrow A = B$ or $B \neq 0$ Counterexample: $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$.

$(AB = 0)$

More generally, $AB = AC \Rightarrow B \neq C$. Counterexample: $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ $C = \begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix}$

"the $k \times k$ identity matrix"

(e). Define $I_k = \mathrm{diag}_k(1, 1, \cdots, 1) = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$. id.

Then for any $m \times n$ matrix A, we have $I_m \cdot A = A = A I_n$.

<u>Q:</u> Is it true that $(AB)C = A(BC)$ whenever all the products make sense?

Answer: Yes — matrix mult is associative.

2. Motivation and associativity of matrix mult.

We'll prove two facts:

(a). Matrix mult. is compatible with composition of linear maps.

(b). Mat. mult is associative

e.g. $\mathbb{R}^2 \to \mathbb{R}^3$ $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \\ x+y \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$

(a) matrix mult vs composition of linear maps.

Recall that any linear maps $T: \mathbb{R}^p \to \mathbb{R}^n$ and $S: \mathbb{R}^n \to \mathbb{R}^m$ have standard matrices A $(n \times p)$ and B $(m \times n)$, respectively.

Now, the composition $S \circ T: \mathbb{R}^p \to \mathbb{R}^n \to \mathbb{R}^m$ $(\mathbb{R}^p \to \mathbb{R}^m)$ makes sense (and is linear, hence $S \circ T$ has a standard $m \times p$ matrix.)

On the other hand, BA is an $m \times p$ matrix.

|| fact
BA

Thm. Let $T: \mathbb{R}^p \to \mathbb{R}^n$ and $S: \mathbb{R}^n \to \mathbb{R}^m$ be linear maps and let A, B be the standard matrices of T, S, respectively. ($M_T = A$, $M_S = B$). Then the standard matrix of $S \circ T$ is BA,

i.e. $M_{S \circ T} = M_S \cdot M_T$.

E.g. Take $m = n = p = 2$. Consider the geometric linear maps

$T: \mathbb{R}^2 \to \mathbb{R}^2$, $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x \\ -y \end{bmatrix}$. $\longrightarrow$ ref. w.r.t. x-axis. $M_T = A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$S: \mathbb{R}^2 \to \mathbb{R}^2$, $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} y \\ x \end{bmatrix}$ $\longrightarrow$ ref. w.r.t. $y = x$, $M_S = B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Let's get the matrix $M_{S \circ T}$ of $S \circ T$ by computing

$S \circ T(e_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $S \circ T(e_2) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. So $M_{S \circ T} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Meanwhile,

$M_S \cdot M_T = BA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = M_{S \circ T}$.

(b). Associativity

Thm: Let A, B, C be matrices such that AB and BC both make sense,

Then $(AB)C = A(BC)$.

① = ② since $R \circ (S \circ T) = (R \circ S) \circ T$. $\square$

Eg.

$$\underset{A}{\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}} \underset{B}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}} \underset{C}{\begin{bmatrix} 0 & -1 \\ -2 & 1 \end{bmatrix}}.$$

$$(AB)C = \begin{bmatrix} 4 & 6 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -12 & 2 \\ 4 & -1 \end{bmatrix}$$

$$\parallel$$

$$A(BC) = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -4 & 1 \\ -8 & 1 \end{bmatrix} = \begin{bmatrix} -12 & 2 \\ 4 & -1 \end{bmatrix}$$

Pf: Say A is $m \times n$, B is $n \times p$, C is $p \times q$. Consider the maps

$R: \mathbb{R}^n \to \mathbb{R}^m$, $S: \mathbb{R}^p \to \mathbb{R}^n$, $T: \mathbb{R}^q \to \mathbb{R}^p$ given by

$R(v) = Av$, $S(w) = Bw$, $T(u) = Cu$. **Fact:** $M_R = A$, $M_S = B$, $M_T = C$.

So $(AB)C = (M_{R \circ S}) \cdot M_T = \underbrace{M_{(R \circ S) \circ T}}_{①}$, $A(BC) = M_R \cdot (M_{S \circ T}) = \underbrace{M_{R \circ (S \circ T)}}_{②}$.