Math 2130. Lecture 13.

02.15.2021.

Last time: · Geometric linear transformations in $\mathbb{R}^2$:

reflections, projections, expansions / contractions, shearing, rotations

Done with Ch. 1!

Today. 2.1. Matrix operations

Notation: (A_{ij}) Given an $m \times n$ matrix A, we use A_{ij} to denote the entry on the ith row, jth col of A for all $1 \le i \le m$, $1 \le j \le n$.

1. Addition and scalar multiplication

- Given two matrices A, B of the same shape, say both $m \times n$, we define their sum to be the $m \times n$ matrix $A+B$ with $(A+B)_{ij} = A_{ij} + B_{ij}$.

- Given a matrix A (say, $m \times n$) and a scalar $c \in \mathbb{R}$. We define the scalar multiple cA to be the matrix with $(cA)_{ij} = cA_{ij}$.

E.g. Let $A = \begin{bmatrix} 3 & 2 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}$. $B = \begin{bmatrix} 6 & 1 \\ 7 & -10 \\ -1 & 5 \end{bmatrix}$. Compute $5A$, $3A-B$

$$5 \cdot A = \begin{bmatrix} 15 & 10 \\ -5 & 5 \\ 0 & 5 \end{bmatrix}, \quad 3A - B = \begin{bmatrix} 9 & 6 \\ -3 & 3 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 6 & 1 \\ 7 & -10 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ -10 & 13 \\ 1 & -2 \end{bmatrix}.$$

Theme of the section: We'll be interested in the properties of matrix operations and the interactions of diff. operations.

Prop. Let A, B, C be matrices of the same shape. Let $r, s \in \mathbb{R}$ be arbitrary scalars. Then

$$[r(A+B)]_{ij} = r[(A+B)]_{ij} = r([A]_{ij} + [B]_{ij})$$
$$= r[A]_{ij} + r[B]_{ij}$$
$$= [rA]_{ij} + [rB]_{ij}$$

(1) $A + B = B + A$

("addition is commutative")

(2) $(A+B)+C = A + (B+C)$

(addition is associative)

(3). $A + 0 = A$

0: zero matrix

(4). $r(A+B) = rA + rB$

(scalar mult. dist. over mat. addition)

(5). $(r+s)A = rA + sA$

(scalar mult. dist. over scalar addition)

(6). $r(sA) = (rs)A$

Pf: These all hold because (i) they hold for numbers (ii) the mat. ops are coordinatewise.

E.g. $r(A+B) = rA + rB$.

$$r = 2, \quad A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 5 & 0 \end{bmatrix}.$$

$$LHS = 2\left(\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 5 & 0 \end{bmatrix} \right) = 2 \begin{bmatrix} 2+1 & 1+0 \\ -1+5 & 3+0 \end{bmatrix} = \begin{bmatrix} 2(2+1) & 2\cdot(1+0) \\ 2\cdot(-1+5) & 2\cdot(3+0) \end{bmatrix}$$

// the entries at each position are equal

$$RHS = 2 \cdot \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} + 2 \cdot \begin{bmatrix} 1 & 0 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} 2\cdot 2 & 2\cdot 1 \\ 2\cdot(-1) & 2\cdot 3 \end{bmatrix} + \begin{bmatrix} 2\cdot 1 & 2\cdot 0 \\ 2\cdot 5 & 2\cdot 0 \end{bmatrix} = \begin{bmatrix} 2\cdot 2 + 2\cdot 1 & 2\cdot 1 + 2\cdot 0 \\ 2\cdot(-1) + 2\cdot 5 & 2\cdot 3 + 2\cdot 0 \end{bmatrix}$$

2. Matrix multiplication.

Recall that

- (inner product) For any row vector $\vec{r} = [a_1 \ a_2 \ \cdots \ a_n]$ and col vec $\vec{c} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ of the same length, we defined their inner product to be the number $\vec{r}\cdot\vec{c} = a_1b_1 + a_2b_2 + \cdots + a_nb_n$.

- (matrix-vector product) Given an $m\times n$ matrix $A = \begin{bmatrix} \vec{r}_1 \\ \vdots \\ \vec{r}_m \end{bmatrix}$ and a vector $\vec{c} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, then the product $A\vec{c}$ equals.

$$A\vec{c} = \begin{bmatrix} \vec{r}_1\cdot\vec{c} \\ \vdots \\ \vec{r}_m\cdot\vec{c} \end{bmatrix}.$$

e.g. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1\cdot(-1) + 2\cdot 0 + 3\cdot 2 \\ 4\cdot(-1) + 5\cdot 0 + 6\cdot 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$

Def. Let A, B be matrices. If # cols of A = # rows of B, then we define the product AB as follows: Say A is $m \times n$ and B is $n \times p$.

(1) (via mat-vec products) write $B = \left[\vec{c}_1 \middle| \vec{c}_2 \middle| \cdots \middle| \vec{c}_p\right]$ and take

$$AB = \left[A\vec{c}_1 \middle| A\vec{c}_2 \middle| \cdots \middle| A\vec{c}_p\right]$$

(2) (via row-col inner products) equivalently, we define AB to be the matrix s.t.

$$(AB)_{ij} = \vec{r}_i \cdot \vec{c}_j, \quad \text{i.e.,} \quad AB = \begin{bmatrix} r_1 \cdot c_1 & r_1 \cdot c_2 & \cdots & r_1 \cdot c_p \\ r_2 \cdot c_1 & & \cdots & \\ & & \cdots & \\ r_m \cdot c_1 & r_m \cdot c_2 & & r_m \cdot c_p \end{bmatrix}$$

where $A = \begin{bmatrix} - & \vec{r}_1 & - \\ & \vdots & \\ - & \vec{r}_m & - \end{bmatrix}$

(in particular, AB is $\underbrace{m}_{\#\text{rows in } A} \times \underbrace{p}_{\#\text{cols in } B}$)

Eg. $A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

BA is not defined since # cols of $B \neq$ # rows of A.

Continued Eg. $A = \begin{array}{c} \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \end{array} \begin{array}{l} r_1 \\ r_2 \end{array}$ $B = \begin{array}{ccc} c_1 & c_2 & c_3 \\ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$ $-1 \cdot 1 + 2 \cdot 1$

Since # cols of A = # rows of B = 2, AB is defined. Let's compute it:

(1). via matrix · vect product

$$AB = \left[Ac_1 \mid Ac_2 \mid Ac_3 \right] = \left[\begin{array}{c|c|c} 1 & 1 & 1 \\ -1 & 1 & -1 \end{array} \right]$$

‖ pick your preferred method!

(2) via row-col inner products

$$AB = \left[\begin{array}{c|c|c} r_1 \cdot c_1 & r_1 \cdot c_2 & r_1 \cdot c_3 \\ \hline r_2 \cdot c_1 & r_2 \cdot c_2 & r_2 \cdot c_3 \end{array} \right] = \left[\begin{array}{c|c|c} 1 & 1 & 1 \\ \hline -1 & 1 & -1 \end{array} \right].$$

E.g.

- $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1\cdot 1 + 1\cdot 3 & 1\cdot 2 + 1\cdot 4 \\ 1\cdot 1 + 1\cdot 3 & 1\cdot 2 + 1\cdot 4 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 4 & 6 \end{bmatrix}$

Note: matrix mult (for square matrices) is

<u>not</u> defined coordinatewise !

$\cancel{\times}$ $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

- $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & 0 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & -5 \end{bmatrix}$. "$(m \times n), (n \times p) \rightsquigarrow (m \times p)$"

- $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 7 & 5 & 1 \\ 2 & -1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 7 & 5 & 1 \\ -6 & 3 & 6 \end{bmatrix}$

$\operatorname{diag}(2, 1, -3)$

Point: multiplying with $\operatorname{diag}(a_1, a_2, \dots, a_n)$ on the left simply scales Row i by a_i $\forall\, 1 \le i \le m$.

Next time: properties of mult. $\left\{ \begin{array}{l} A(B+C) = AB + AC \checkmark \\ AB = BA \ \times \\ + \text{ more} \dots \end{array} \right.$