Math 2130. Lecture 12. 02.12.2021.

Midterm 1 Date: Monday, March 1.

Last time: (Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear map.)

- (Image / Range) $\operatorname{Im} T = \{ T(\vec{v}) \mid \vec{v} \in \mathbb{R}^n \}$
- (Kernel) $\operatorname{Ker} T = \{ \vec{v} \in \mathbb{R}^n \mid T(\vec{v}) = \vec{0} \}$

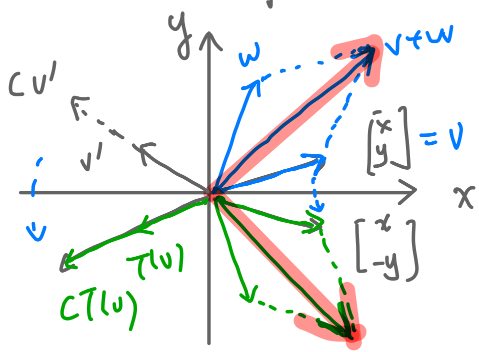
Eff. criteria $\longleftrightarrow$
- (Surjectivity / Onto) $\operatorname{Im} T = \mathbb{R}^m$
- (injectivity / one-to-one) $\operatorname{Ker} T = \{\vec{0}\}$ $\left(\Longleftrightarrow \boxed{T(\vec{v}_1) = T(\vec{v}_2) \text{ only if } \vec{v}_1 = \vec{v}_2} \right.$

Today. · Geometric transformations of $\mathbb{R}^2$

1. Geometric linear maps on $\mathbb{R}^2 \to \mathbb{R}^2$

First example. (reflection across the x-axis)

Consider the map $\mathrm{Ref}_x : \mathbb{R}^2 \to \mathbb{R}^2$, $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x \\ -y \end{bmatrix}$. The map reflects every vec. w.r.t (with respect to) the x-axis.



$$v = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \quad w = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

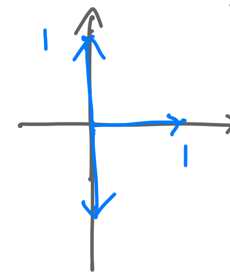
(we'll not mark the vectors with arrows from now on)

· T is linear (in pictures) : Ex: $T(v+w) = T(v) + T(w)$

$$T(cv) = cT(v) \quad \forall c \in \mathbb{R}.$$

· What is the standard matrix of T? $\left(\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x \\ -y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \right)$

$T(e_1)$ $T(e_2)$



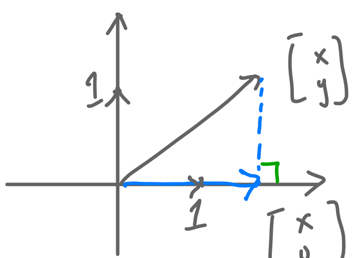
$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. (EF.

· Is T injective? Surjective?

EF(A) has no zero row, so T is surjective.

EF(A) has a pivot in every col, so T is inj.

Example 2. Projection onto the x-axis.



The map $T = \mathrm{Proj}_x : \mathbb{R}^2 \to \mathbb{R}^2$, $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x \\ 0 \end{bmatrix}$

is also a linear map. It's called the projection onto the x-axis. Note that it has standard matrix $\left[T(e_1) \middle| T(e_2) \right] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

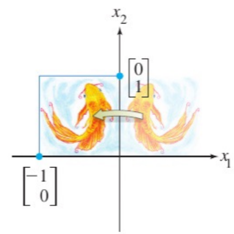
Ex: Show that Proj_x is neither surj nor inj.

More geometric maps:

(a). more reflections:

- refl. w.r.t. the y-axis : $\begin{bmatrix} x \\ y \end{bmatrix} \longmapsto \begin{bmatrix} -x \\ y \end{bmatrix}$

Reflection through the x_2-axis

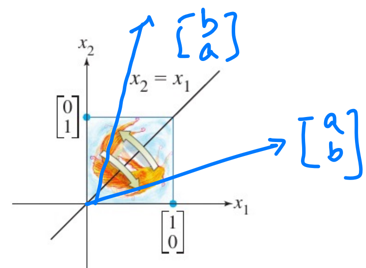


$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow$ standard matrix.

surj. & inj.

- refl. w.r.t. the "diagonal line" $y = x$: $\begin{bmatrix} a \\ b \end{bmatrix} \longmapsto \begin{bmatrix} b \\ a \end{bmatrix}$

Reflection through the line $x_2 = x_1$

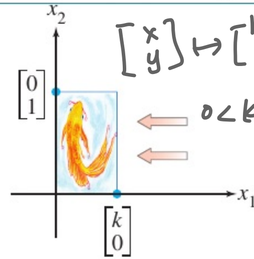
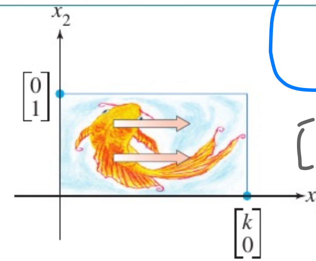
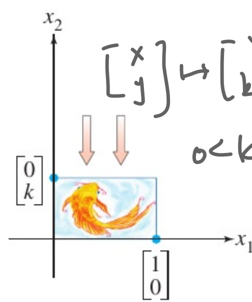
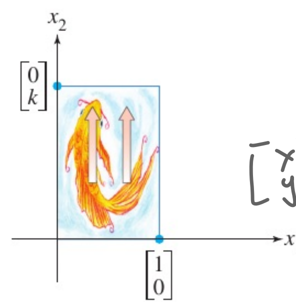


$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \longrightarrow$ standard matrix.

surj & inj.

(b) Contractions and expansions

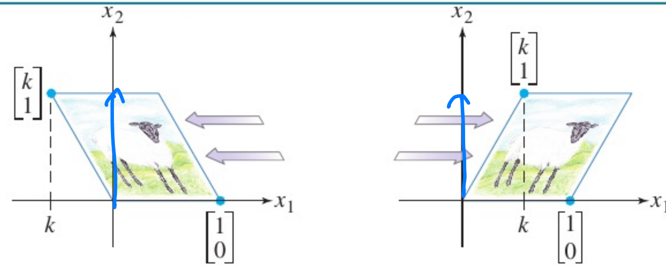
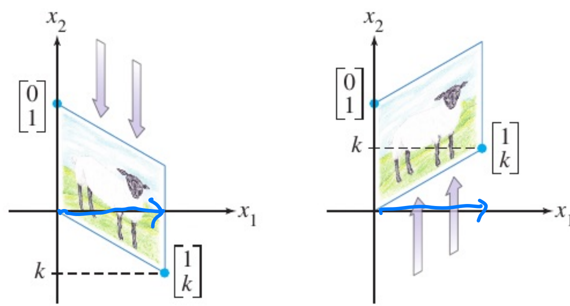
TABLE 2 Contractions and Expansions

Transformation	Image of the Unit Square		Standard Matrix
Horizontal contraction and expansion	 $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} kx \\ y \end{bmatrix}$ $0<k<1$ $0 < k < 1$	 $k > 1$	$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} kx \\ y \end{bmatrix}$ $k>1$
Vertical contraction and expansion	 $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x \\ ky \end{bmatrix}$ $0<k<1$ $0 < k < 1$	 $k > 1$	$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$ $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x \\ ky \end{bmatrix}$ $k>1$.

For every $k>0$, the four maps are all both surj and inj. (since $k \neq 0$)

(c) Shears:

TABLE 3 Shears

Transformation	Image of the Unit Square	Standard Matrix
Horizontal shear	 $k < 0$ $k > 0$	$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
Vertical shear	 $k < 0$ $k > 0$	$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

$T(e_1)$ $\quad T(e_2) = \begin{bmatrix} k \\ 1 \end{bmatrix}$ sheared from $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$T(e_1) = \begin{bmatrix} 1 \\ k \end{bmatrix}$ $\quad T(e_2)$

all surj & inj.

(d). rotations : $\mathrm{Rot}_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ rotating every vector by angle α is a linear map.

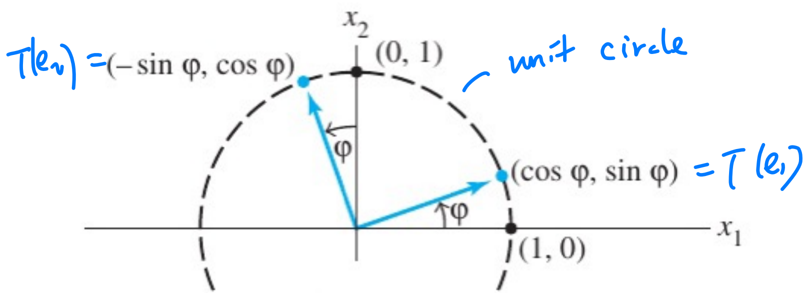


FIGURE 1 A rotation transformation.

standard matrix $= \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$.

Ex: · Convince yourself that Rot_α is a linear map.

· Determine if Rot_α is inj & surj for each angle α.

□