Math 2130. Lecture 11.

02.10.2021

Last time. (a) Examples and formulas of linear vs. non-linear maps

(b) Linear maps $\equiv$ matrix multiplication

Today.

- making (b) precise, as a thm.
- Some terminology for linear maps.
- Some geometric linear maps on $\mathbb{R}^2$.

linear?

$$x \longmapsto 3x - 4.$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \longmapsto \begin{bmatrix} x^2 \\ y-x \\ 2x \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \longmapsto \begin{bmatrix} 2x+y \\ -x-y \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

1. Linear maps $\equiv$ matrix multiplication.

Thm. Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Then there exists a unique $m \times n$ matrix M st. $T(\vec{v}) = A \cdot \vec{v}$. In fact, A is the matrix

$$A = \left[T(e_1) \,\vdots\, T(e_2) \,\vdots\, \cdots \,\vdots\, T(e_n) \right]$$

where $\{e_1, e_2, \cdots, e_n\}$ is the standard basis of $\mathbb{R}^n$.

Def: We call the matrix A the <u>standard matrix</u> of T.

Key Idea: We'll often study a linear map T via its standard matrix.

e.g. $T: \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 2x+y \\ -x-y \end{bmatrix}$

$$= \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

$$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

2. Image, Kernel, Injectivity and Surjectivity

Def. Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear map.

(1) The <u>image</u> of T is defined to be the set

$$\operatorname{Im} T := \{ T(\vec{v}) \mid \vec{v} \in \mathbb{R}^n \} \subseteq \mathbb{R}^m .$$

(2) The <u>kernel</u> of T is the set

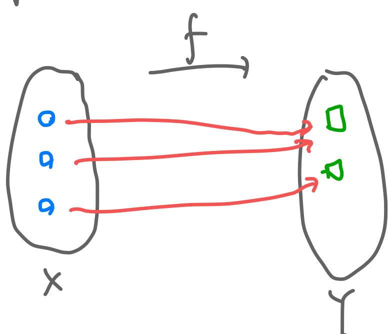
$$\operatorname{Ker} T := \{ \vec{v} \in \mathbb{R}^n \mid T(\vec{v}) = 0 \} \subseteq \mathbb{R}^n .$$

e.g. $T: \mathbb{R}^2 \to \mathbb{R}$, $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto x-y$. $\ker T = \{ \begin{bmatrix} x \\ y \end{bmatrix} : x-y=0 \} = \{ \begin{bmatrix} y \\ y \end{bmatrix} : y \in \mathbb{R} \}$ $\ni \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

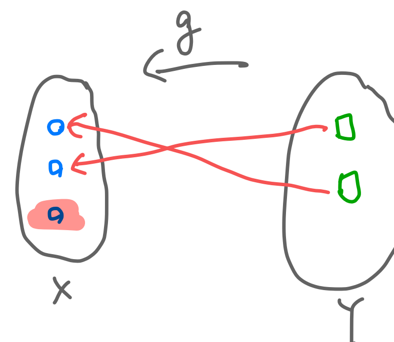
e.g. $S: \mathbb{R}^2 \to \mathbb{R}^2$, $\begin{bmatrix} x \\ y \end{bmatrix} \longmapsto \begin{bmatrix} 2x+y \\ 4x+2y \end{bmatrix}$. Ex: $\begin{bmatrix} 1 \\ -2 \end{bmatrix} \in \operatorname{Ker} T$, $\begin{bmatrix} 1 \\ -3 \end{bmatrix} \notin \ker T$.

(3) We say T is <u>surjective</u> if $\operatorname{Im} T = \mathbb{R}^m$, i.e., if every elt in $\mathbb{R}^m$ is the output of at least one input.

(4) We say T is <u>injective</u> if $T(\vec{v}) = T(\vec{v}\,')$ for $\vec{v}, \vec{v}\,' \in \mathbb{R}^n$ only when $\vec{v} = \vec{v}\,'$, i.e., every elt in $\mathbb{R}^m$ is the output of at most one input.



surj. but not inj.



inj. but not surj

An equivalent def of injectivity.

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear map.

Prop: The linear map T is injective iff $\ker T = \{\vec{0}\}$.

(So some people define an inj. linear map to be a linear map with kernel $\{0\}$.)

Pf: "only if": Suppose T is inj. Take $\vec{v} \in \ker T$. Then $T(\vec{v}) = \vec{0}$. On the other hand, we know that $T(\vec{0}) = \vec{0}$. So $\vec{v} = \vec{0}$. Therefore $\ker T = \{\vec{0}\}$.

"if": Suppose $\ker T = \{\vec{0}\}$. Let $\vec{v}, \vec{v}'$ be such that $T(\vec{v}) = T(\vec{v}')$. Then $T(\vec{v} - \vec{v}') = T(\vec{v}) - T(\vec{v}') = \vec{0}$, so $\vec{v} - \vec{v}' = 0$, i.e., $\vec{v} = \vec{v}'$. Therefore T is inj.

Some translations

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear map and let A be the standard matrix of T.

(e.g. $T: \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 2x+y \\ x-y \\ 3y \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 0 & 3 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix}$

$\parallel$

$x \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$.

(a) (Image vs. Span Membership)

For a fixed $\vec{w} \in \mathbb{R}^m$, we have

$\vec{w} \in \operatorname{Im} T$

e.g. Is $\begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix} \in \operatorname{Im} T$?

$\Longleftrightarrow$ $\vec{w} \in$ Span of the columns of A.

$\downarrow$

Ex: Check $\left[\begin{array}{cc|c} 2 & 1 & 6 \\ 1 & -1 & 2 \\ 0 & 3 & 1 \end{array}\right]$.

$\overset{\text{before}}{\Longleftrightarrow}$ EF$\left(\left[A \mid \vec{w} \right] \right)$ has no row of the form $\begin{bmatrix} 0 & 0 & \cdots & 0 & b \end{bmatrix}$ where $b \neq 0$.

(b) (Surjectivity vs. spanning property)

We have

T is surjective, i.e., $\operatorname{Im} T = \mathbb{R}^m$

$\iff$ the columns of A span all of $\mathbb{R}^m$

$\iff$ $EF(A)$ has no zero rows.

e.g. $T: \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 2x+y \\ x-y \\ 3y \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 0 & 3 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix}$

$\parallel$

$x \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$.

e.g. $\mathbb{R}^m = \mathbb{R}^3$.

$\#$ cols of $A = 2 < 3$.

So the cols of A can't span $\mathbb{R}^3$.

So T can't be surj.

Ex. $T: \mathbb{R}^2 \to \mathbb{R}^2$, $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x-y \\ 2x-2y \end{bmatrix}$.

(c) (Injectivity vs. linear independence)

We have

T is inj, i.e., $\ker T = \{\vec{0}\}$

$\Longleftrightarrow$ the only lin comb. of the cols of A that equals $\vec{0}$ is the trivial one, i.e., the columns of A are linearly ind.

$\Longleftrightarrow$ $EF(A)$ has a pivot in every column.

e.g. $T: \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 2x+y \\ x-y \\ 3y \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 0 & 3 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix}$

$= x\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}.$

e.g. $\begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 0 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 0 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 3 \\ 0 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}$

all cols pivot $\xrightarrow{E.X.}$ T is inj.

Ex. $T: \mathbb{R}^3 \to \mathbb{R}^3$, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} x+2y \\ 3x+6y \\ 0 \end{bmatrix}$.

Is it surj / inj ?

Size bounds for surjectivity / injectivity :

Ex.: Using the equivalences on the previous two pages.

Show that if $T: \mathbb{R}^n \to \mathbb{R}^m$ is linear, then

(1). if T is surj, then $n \geq m$.

(2). if T is inj, then $n \leq m$.