Math 2130. Lecture 10. 02.08.2021.

Last time:

- Size bounds on spanning / lin. ind. sets of $\mathbb{R}^n$.
- def. of linear maps: a map $T: \mathbb{R}^n \to \mathbb{R}^m$ is <u>linear</u> if

$$\text{linearity} \begin{cases} (1) & T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2) \quad \forall \vec{v}_1, \vec{v}_2 \in \mathbb{R}^n \\ (2) & T(c \cdot \vec{v}) = c \cdot T(\vec{v}) \quad \forall c \in \mathbb{R}, \vec{v} \in \mathbb{R}^n. \end{cases}$$

implies

$$T\underbrace{(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k)}_{\text{new output}} = c_1 \underbrace{T(\vec{v}_1)} + c_2 \underbrace{T(\vec{v}_2)} + \cdots + c_k \underbrace{T(\vec{v}_k)}_{\text{known outputs}}$$

for all $c_1, c_2, \ldots, c_k \in \mathbb{R}$, $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^n$.

useful for producing new outputs from old ones

An important spanning set of $\mathbb{R}^n$: $\left\{ e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right\}$

Today. · Testing linearity / Formulas of linear maps

· "Linear maps are the same as matrix multiplications"

0. Linear maps must send $\vec{0}$ to $\vec{0}$.

Prop. 0: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Then $T(\vec{0}) = \vec{0}$.

$\boxed{a = 2a \Rightarrow 2a - a = 0 \Rightarrow a = 0}$

pf: Since $\vec{0} = \vec{0} + \vec{0}$ in $\mathbb{R}^n$, we have

$$T(\vec{0}) = T(\vec{0} + \vec{0}) \overset{\text{lin.}}{=\!=} T(\vec{0}) + T(\vec{0}) = 2\,T(\vec{0}) \quad (*)$$

$$\Rightarrow T(\vec{0}) = \vec{0} \text{ in } \mathbb{R}^m.$$

(e.g. $m = 3$. $T(\vec{0}) = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

$(*)$: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}$.)

E.g. (Diff) We saw that the formal diff map

$D: P_3 \to P_2$, $at^3 + bt^2 + ct + d \longmapsto 3at^2 + 2bt + c$ is linear.

Note that $D(\vec{0}) = \vec{0}$.

1. Formulas of linear maps

Examples. Determine if the following maps $T: \mathbb{R}^n \to \mathbb{R}^m$ is linear.

(1). $T: \mathbb{R} \to \mathbb{R}$, $T(x) = 2x \quad \forall x \in \mathbb{R}$ $\quad (x \mapsto 2x)$

Let's check the axioms.

(i) $T(x+y) = 2(x+y) = 2x+2y$

$T(x) + T(y) = 2x+2y$

$\Bigg\} \Rightarrow T(x+y) = T(x) + T(y) \quad \forall x, y \in \mathbb{R}.$

(ii) $T(c \cdot x) = 2cx$

$cT(x) = c \cdot (2x) = 2cx$

$\Bigg\} \Rightarrow T(cx) = cT(x) \quad \forall c \in \mathbb{R}, x \in \mathbb{R}.$

By (i), (ii), T is linear.

12). $T: \mathbb{R} \to \mathbb{R}$, $x \mapsto 2x+3$.

Method 1. Check additive property:

(i) $T(x+y) = 2(x+y)+3 = 2x+2y+3$

$T(x)+T(y) = (2x+3)+(2y+3) = 2x+2y+6$ } they are not equal,

So T doesn't respect addition. therefore T is not linear.

Method 2. $T(0) = 2\cdot 0+3 = 3 \neq 0$, so T is not linear by Prop. 0.

Rmk: It's an unfortunate fact that people often call a function $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto ax+b$ a linear function for all constants a, b, but f is a linear transformation in the lin. alg. sense only if $b=0$.

(3) $T: \mathbb{R} \to \mathbb{R}$ $\quad x \longmapsto x^2$ Ex: Show that T also violates (ii).

Try Prop o? $T(0) = 0^2 = 0$. So we can't quickly conclude that T is not linear.

Additive property?

$$T(x+y) = (x+y)^2 = x^2 + y^2 + 2xy$$

$$T(x) + T(y) = x^2 + y^2$$

not ~~always~~ equal. eg. when $x = y = 1$,

So T doesn't respect addition, therefore it's not linear.

Another way to write the soln: (Use a counter-example.) For $x = y = 1$, we have $T(x+y) = T(1+1) = T(2) = 4$ while $T(x) + T(y) = x^2 + y^2 = 1^2 + 1^2 = 2$, so $T(x+y) \neq T(x) + T(y)$ and T is not linear.

(f) $T: \mathbb{R} \to \mathbb{R}^2$, $x \mapsto \begin{bmatrix} 2x \\ x^2 \end{bmatrix} = \begin{bmatrix} T_1(x) \\ T_2(x) \end{bmatrix}$ $T_1(x)$ — linear, $T_2(x)$ — not linear

Soln:

$$T(x+y) = \begin{bmatrix} 2(x+y) \\ (x+y)^2 \end{bmatrix} = \begin{bmatrix} 2x+2y \\ x^2+y^2+2xy \end{bmatrix} \checkmark$$

not equal

$$T(x)+T(y) = \begin{bmatrix} 2x \\ x^2 \end{bmatrix} + \begin{bmatrix} 2y \\ y^2 \end{bmatrix} = \begin{bmatrix} 2x+2y \\ x^2+y^2 \end{bmatrix} \checkmark$$

Since $\underbrace{(x+y)^2 \neq x^2+y^2 \text{ in general}}_{T_2(x) \text{ is not linear}}$, $T(x+y) \neq T(x)+T(y)$ in general.

Therefore T is not linear.

Intuition: "exponents" are bad.

(5). $T: \mathbb{R}^3 \to \mathbb{R}^3$, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \longmapsto \begin{bmatrix} 3x-y \\ y+z \\ 2x-z \end{bmatrix}$.

Additive property: Ex.

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}\right) = T\left(\begin{bmatrix} x+x' \\ y+y' \\ \cdots \end{bmatrix}\right) = \begin{bmatrix} 3(x+x')-(y+y') \\ \cdots \\ \cdots \end{bmatrix}$$

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) + T\left(\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}\right) = \begin{bmatrix} 3x-y \\ \vdots \\ \vdots \end{bmatrix} + \begin{bmatrix} 3x'-y' \\ \vdots \end{bmatrix} = \begin{bmatrix} 3x-y+3x'-y' \\ \vdots \end{bmatrix}$$

Scaling:

$$T\left(c\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = T\left(\begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}\right) = \begin{bmatrix} 3cx-cy \\ cy+cz \\ 2cx-cz \end{bmatrix}$$

$$cT\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = c\begin{bmatrix} 3x-y \\ y+z \\ 2x-z \end{bmatrix} = \begin{bmatrix} 3cx-cy \\ cy+cz \\ 2cx-cz \end{bmatrix}$$

)).

Since T respects add. and scalar mult, T is linear.

Ex: The map $T_1: \mathbb{R}^2 \to \mathbb{R}^3$, $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 2x \\ 3x-y \end{bmatrix}$ is linear,

while the map $T_2: \mathbb{R}^2 \to \mathbb{R}^3$, $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} \sin x \\ 2y \\ 2\pi xy \end{bmatrix}$ is not.

Note: Every map $T: \mathbb{R}^n \to \mathbb{R}^m$ must send an input $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ to an output of the form $\begin{bmatrix} T_1(x_1, \dots, x_n) \\ \vdots \\ T_m(x_1, \dots, x_n) \end{bmatrix}$ $\to$ $T_1, T_2, \dots, T_m$ are the "component maps" and together they are equivalent to T.

Conjecture: T is linear $\Longleftrightarrow$ $T_1, T_2, \dots, T_m$ all take linear comb. of the entries in the input vector.

2. Linear maps $\equiv$ matrix multiplication

Fact 1. (that we can already prove) "Matrix multiplication are linear maps".

Let M be an $m \times n$ matrix. Then the map $T_M : \mathbb{R}^n \to \mathbb{R}^m$, $\vec{v} \mapsto M \cdot \vec{v}$

(e.g. $M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$) (e.g. $T_M : \mathbb{R}^3 \to \mathbb{R}^2$ $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. $= \begin{bmatrix} x + 2y + 3z \\ 4x + 5y + 6z \end{bmatrix}$)

By the properties of matrix-vector products:

(i) $T_M(\vec{v} + \vec{w}) = \boxed{M(\vec{v} + \vec{w}) = M \cdot \vec{v} + M \cdot \vec{w}} = T_M(\vec{v}) + T_M(\vec{w})$

(ii) $T_M(c\vec{v}) = \boxed{M(c\vec{v}) = c(M \cdot \vec{v})} = c\, T_M(\vec{v})$

By (i), (ii), T_M is linear.

Fact 2. (New) "Linear maps are all matrix multiplications".

E.g. $T: \mathbb{R}^2 \to \mathbb{R}^2$, $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 2x-y \\ 3x \end{bmatrix}$ is linear (Ex: prove it);

Note that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$, so T is "multiplication by the matrix $\begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}$".

$\downarrow$

Thm. This generalizes. We'll have the precise statement next time.