Math 2130. Linear Algebra

Course Information.

Instructor: Tianyuan Xu (Eddy)

Website: https://math.colorado.edu/~tixu6187/2130.html

— has Canvas link

— lecture notes & HW posted under "LECTURES" tab

Office Hours: Mondays. 11 am - noon, + appointments.

Grading: HW 20%, Midterm 20% × 2, Final 40%

HW: — to be submitted on Canvas/Assignments, pdf only

— due on Wednesday nights 11:59 pm.

(the deadline is strict; no late submission possible)

— HW1 due on Jan 27.

— posted the previous Wed. Jan 20.

Textbook: "Linear Algebra and its Applications, Fifth Edition"

by Lay, Lay and McDonald.

— available in Canvas/Files.

On to the math.

Today. Systems of linear equations. Matrices. Row operations.

1. Definitions and motivations.

Def. By a system of linear equations (L.E.S.) we mean a finite set of equations of the form

$$(*)\quad \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \quad\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

where a_{ij} is a constant $\forall\, 1 \le i \le m,\ 1 \le j \le n$ and $b_1, \ldots, b_m$ are constants.

We may encode an SEL of the form

$$(*)\quad \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

eg.

$$\begin{cases} 2x + y = 3 \\ y - x = 4 \end{cases}$$

$$\Updownarrow$$

$$\begin{array}{c} \begin{bmatrix} 2 & 1 & 3 \\ -1 & 1 & 4 \end{bmatrix} \\ \begin{array}{ccc} x & y & \end{array} \end{array}$$

by the <u>matrix</u>. (a rectangular array)

$$A_* = \begin{array}{c} \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ & \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right] \\ \begin{array}{cccc} x_1 & x_2 & \cdots & x_n \end{array} \end{array}$$

rows $\leftrightarrow$ equations

cols $\leftrightarrow$ variables

<u>Conversely</u>, we can (and will) recover the system from its matrix.

Def: (1) (Solutions) A <u>solution</u> of a linear system of the form $*$ is just a tuple of values $(x_1, x_2, \ldots, x_n)$ for which all the equations hold.

(2). (Consistency) We say a linear equation system is <u>consistence</u> if it has at least one soln and <u>inconsistent</u> otherwise.

e.g. $\begin{cases} x+y=3 \\ 2x+2y=5 \end{cases}$ is inconsistent ; $\begin{cases} x+y=3 \\ y=1 \end{cases}$ is consistent with exactly one soln $(2,1)$;

$\{x-y=1$ is consistent and has an infinite soln set $\{(t+1, t) : t \text{ is a constant}\}$

(3) (Equivalence of SELs) Two linear equation systems are <u>equivalent</u> if they have the same soln set. **eg.** $\begin{cases} x+y=3 \\ y=1 \end{cases} \iff \begin{cases} x=2 \\ y=1 \end{cases}$

Central question: Given a linear equation (system $*$), how can we tell if it's consistent? If it is, how can we find all its solns?

Fact: (Trichotomy) A linear equation system always has 0, 1, or infinitely many solns.

We'll try to answer the central question in terms of the matrix encoding $*$), by using matrix manipulations.

2. Row operations on matrices.

Elimination of variables. Recall that we can solve linear equation systems by eliminating variables. We'll translate the method to matrices.

Eg.1.
$$\begin{cases} x - 3y = 1 & ① \\ 2x = 4 & ② \end{cases}$$

$① \leftrightarrow ②$

$$\begin{cases} 2x = 4 & ①' \\ x - 3y = 1 & ②' \end{cases}$$

$①' \times \frac{1}{2}$

$$\begin{cases} x = 2 & ①'' \\ x - 3y = 1 & ②'' \end{cases}$$

$$\begin{bmatrix} 1 & -3 & 1 \\ 2 & 0 & 4 \end{bmatrix}$$

switch two rows.

$$\begin{bmatrix} 2 & 0 & 4 \\ 1 & -3 & 1 \end{bmatrix}$$

multiply (every entry in) a row by a nonzero constant.

$$\begin{bmatrix} 1 & 0 & 2 \\ 1 & -3 & 1 \end{bmatrix}$$

$$\begin{cases} x = 2 & ①'' \\ x - 3y = 1 & ②'' \end{cases}$$

"eliminate x from ②''"

$$\begin{cases} x = 2 & ①''' \\ -3y = -1 & ②''' = ②'' - ①'' \end{cases}$$

scale an equation

$$\begin{cases} x = 2 \\ y = \frac{1}{3} \end{cases}$$

DONE.

$$\begin{array}{c} R1 \\ R2 \end{array} \begin{bmatrix} 1 & 0 & 2 \\ 1 & -3 & 1 \end{bmatrix} \to \text{nice : } x = 2$$

subtract a multiple of a row (Row 1) from another row (Row 2)

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & -3 & -1 \end{bmatrix}$$

scale a row by a nonzero constant

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & \frac{1}{3} \end{bmatrix} \to \text{nice: } y = \frac{1}{3}.$$

E.g. 2.

$$\begin{cases} 2x - 3y = 1 & ① \\ 4x + y = 23 & ② \end{cases}$$

$① \times 2 \downarrow$

$$\begin{cases} 4x - 6y = 2 & \text{"}①\text{"} \\ 4x + y = 23 & \text{"}②\text{"} \end{cases}$$

$② - ① \downarrow$

$$\begin{cases} 4x - 6y = 2 & ① \\ 7y = 21 & ② \end{cases}$$

$② \times \frac{1}{7} \downarrow$

$$\begin{cases} 4x - 6y = 2 & ① \\ y = 3 & ② \end{cases}$$

$$\begin{array}{c} R1 \\ R2 \end{array} \begin{bmatrix} 2 & -3 & 1 \\ 4 & 1 & 23 \end{bmatrix}$$

"scale"

$$\begin{bmatrix} 4 & -6 & 2 \\ 4 & 1 & 23 \end{bmatrix}$$

can be combined

$\boxed{R_2 - 2 \cdot R_1}$

"Subtract."

$$\begin{bmatrix} 4 & -6 & 2 \\ 0 & 7 & 21 \end{bmatrix}$$

"scale"

$$\begin{bmatrix} 4 & -6 & 2 \\ 0 & 1 & 3 \end{bmatrix}$$

Legal operations.

- $R_i \leftrightarrow R_j$
- $R_i \to c \cdot R_i$, $c \neq 0$
- $R_i \to R_i + c \cdot R_j$.

$$\begin{cases} 4x - 6y = 2 & ① \\ y = 3 & ② \end{cases}$$

① + ② × 6

$$\begin{cases} 4x = 20 & ① \\ y = 3 & ② \end{cases}$$

① × $\frac{1}{4}$

$$\begin{cases} x = 5 \\ y = 3 \end{cases}$$

$$\begin{array}{c l} \begin{bmatrix} 4 & -6 & 2 \\ 0 & 1 & 3 \end{bmatrix} & \begin{array}{l} R_1 \\ R_2 \end{array} \end{array}$$

"replacement"

$R_i \to R_i + c \cdot R_j$

$$\begin{array}{c l} \begin{bmatrix} 4 & 0 & 20 \\ 0 & 1 & 3 \end{bmatrix} & \begin{array}{l} R_1 + 6 \cdot R_2 \\ R_2 \end{array} \end{array}$$

scale

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \end{bmatrix} \to \begin{cases} x = 5 \\ y = 3. \end{cases}$$

Eg. 3.

$$\begin{cases} 3z = 9 \\ 2x - z = 5 \\ 2y + z = 1 \end{cases}$$

Ex. Try to solve the above system and write down the corresponding operations on matrices.

Next time: We'll define (legal) row operations carefully and explain how to effectively use them to solve linear systems.