MATH 2001. SOLUTIONS TO COUNTING PROBLEMS
(permutations and combinations)

(1) How many lists of lengths 3 can be made from the symbols A, B, C, D, E, F if ...

(a) ...repetition is allowed?

There are $6^3 = 216$ such lists.

(b) ...repetition is not allowed?

There are $6 \times 5 \times 4 = 120$ such lists.

(c) ...repetition is allowed and the list must start with the letter A?

There are $6^2 = 36$ such lists, since there are this many possibilities for what the last two entries of the list are.

(d) ...repetition is not allowed and the list must not start with the letter A?

There are $5 \times 4 = 20$ such lists, since there are this many possibilities for what the last two entries of the list are.

(e) ...repetition is allowed and the list must contain the letter A?

There are $6^3 = 216$ lists in total when repetition is allowed, and of those $5^3 = 125$ do not use A (equivalently, they use only the letters B, C, D, E or F), so the desired number is 216-125=91.

(f) ...repetition is not allowed and the list must contain the letter A?

There are $6 * 5 * 4 = 120$ lists in total when repetition is not allowed, and of those $5 \times 4 \times 3 = 60$ do not use A (equivalently, they use only the letters B, C, D, E or F), so the desired number is 120-60=60.

(2) Consider lists made from the symbols A, B, C, D, E, with repetition allowed.

(a) How many such length-4 lists have at least one letter repeated?

There are $5^4 = 625$ length-4 lists in total, of which $5 \times 4 \times 3 \times 2 = 120$ have no repeated letter, so the desired number is $625 - 120 = 505$.

(b) How many such length-4 lists contain exactly two different letters?

e can specify such a list by first specifying which two letters will be used; this can be done in $\binom{5}{2} = 10$ ways. After this, say, the two letters used are x and y, it suffices to specify which of the four positions in the list has x in it. This amounts to specifying a subset of $\{1, 2, 3, 4\}$ (the positions) that is not empty (so x indeed appears) and not equal to $\{1, 2, 3, 4\}$ (so y appears), which can be done in $2^4 - 2 = 14$ ways. It follows that the desired number is $10 \times 14 = 140$.

(c) How many such length-5 lists have at least one letter repeated?

The same logic used in Part (a) applies, and the answer is $5^5 - 5 \times 4 \times 3 \times 2 \times 1 = 3005$.

(d) How many such length-5 lists contain exactly two different letters?

The same logic used in Part (b) applies, and the answer is $\binom{5}{2} \times (2^5 - 2) = 300$.

(e) How many such length-6 lists have at least one letter repeated?

With only 5 letters to choose from, length-6 lists all have repetitions, so we just need to count all the length-6 lists, of which there are 6^5.

(f) How many such length-5 lists are there if the letters A, B, C must appear consecutively in the list?

For each $i \in \{1, 2, 3\}$, let X_i be the set of all the length-5 list whose i-th, $(i+1)$-th, and $(i+2)$-th letters are A, B, and C, respectively. Then the desired number is $|X_1 \cup X_2 \cup X_3|$. Note that the sets X_1, X_2, X_3 are pairwise disjoint, because everything in X_1 has C as the third letter, everything in X_2 has B as the third letter, and everything in X_3 has A as the third letter. It follows that $|X_1 \cup X_2 \cup X_3| = |X_1| + |X_2| + |X_3| = 5^2 + 5^2 + 5^2 = 75$.

(g) How many such length-6 lists are there if the letters A, B, C must appear consecutively in the list?

We can set up the sets X_i's as in the previous part, but allowing $i = 4$ this time since we have length-6 lists. We can then use the inclusion-exclusion principle to count the desired number $|X_1 \cup X_2 \cup X_3 \cup X_4| = |X_1| + |X_2| + |X_3| + |X_4| - |X_1 \cap X_2| - \dots$, where the only non-empty intersection of multiple sets is $X_1 \cap X_4 = \{ABCABC$ (the only list with two ABC's)$\}$. It follows that $|X_1 \cup X_2 \cup X_3 \cup X_4| = 4 * 5^3 - 1 = 499$.

(h) How many such length-5 lists are there if the list must start with a vowel, end with a vowel, and use all the five letters?

The length-5 lists using all the five letters are precisely all the length-5 lists with no repetition allowed. For such lists, to specify a list starting and ending with a vowel it suffices to specify what the beginning vowel is (two choices, A or E) and then how the middle three letters are arranged ($3 \times 2 \times 1 = 6$ choices), so the desired number is $2 \times 6 = 12$.