Math 2001. Solutions to Counting Problems, Part 2
(the inclusion-exclusion principle, and problems involving multisets)

(1) State the inclusion-exclusion principle for two sets (the one computing the cardinality of $A \cup B$ for sets A, B).

The principle says that $|A \cup B| = |A| + |B| - |A \cap B|$.

(2) At a certain university 523 of the seniors are history majors or math majors (or both). There are 100 senior math majors, and 33 seniors are majoring in both history and math. How many seniors are majoring in history?

Let A and B be the sets of math majors and history majors, respectively. Then by the principle from (1), the number of history majors is

$$|A| = |A \cup B| + |A \cap B| - |B| = 523 + 33 - 100 = 456.$$

(3) State the inclusion-exclusion principle for three sets (the one computing the cardinality of $A \cup B \cup C$ for sets A, B, C).

The principle says that

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

(4) How many 7-digit binary strings begin in 1 or end in 1 or have exactly four 1's?

Let A, B and C be the sets of 7-digit binary string that begin in 1, end in 1, and have exactly four 1's, respectively. Then $|A| = |B| = 2^6 = 64, |C| = \binom{7}{4} = 35, |A \cap C| = |B \cap C| = \binom{6}{3} = 20, |A \cap B| = \binom{5}{2} = 10$, and $\cap A \cap B = 2^5$. It follows from the principle from (3) that the desired number is

$$|A \cup B \cup C| = 64 + 64 + 35 - 32 - 20 - 20 + 10 = 101.$$

(5) How many 10-element multisets can be made from the symbols 1, 2, 3, 4?

This can be solved with the bars-and-stars method, with 4-1=3 bars separating 10 stars representing the elements of the multiset into 4 regions. It follows that the answer is $\binom{10+4-1}{4-1} = \binom{13}{3} = 286$.

(1) A bag contains 20 identical red balls, 20 identical blue balls and 20 identical green balls. You reach in and grab 15 balls. How many different outcomes are possible?

This can be solved with the bars-and-stars method as well, with 3-1=2 bars separating the stars (standing for the balls) into 3 regions (corresponding to the colors), so the number of possible outcomes is $\binom{15+3-1}{3-1} = \binom{17}{2} = 136$.

(2) A bag contains 20 identical red balls, 20 identical blue balls, 20 identical green balls, and one white ball. You reach in and grab 15 balls. How many different outcomes are possible?

There are two kinds of outcomes, those that contains the unique white ball and those that do not. The latter involve only the three other colors, so there are $\binom{15+3-1}{3-1} = \binom{17}{2} = 136$ of them. The outcomes containing the white ball correpond naturally and bijectively to the configurations of 14 balls using the red, blue or green colors, so there are $\binom{14+3-1}{3-1} = \binom{16}{2} = 120$ of them. It follows that the desired answer should be $136 + 120 = 256$.

(3) How many length-6 lists can be made from the symbols a, b, c, d, e, f, and g, if repetition is allowed and the list is in alphabetical order? (Examples: bbcegg, but not bbbagg.)

Given the requirement that the list is in alphebetical order, to specify such a list is just to specify a multiset of size 6 whose elements are from the six symbols a, b, c, d, e, f. For example, the multiset [b,c,d,a,a,b] correspond to the list aabbcd. Such multisets can be enumerated by the bars-and-stars method, which implies that the answer should be $\binom{6+6-1}{6-1} = \binom{11}{5}$.

(4) How many lists (x, y, z) of three nonnegative integers are there with $x + y + z = 100$?

This can be solved with the bars-and-stars methods, with 3-1=2 bars separating 100 stars into 3 regions corresponding to x, y and z. It follows that the answer should be $\binom{100+3-1}{3-1} = \binom{102}{2} = 5151$.

(5) How many lists (x, y, z) of three nonnegative integers are there with $x \leq y \leq z \leq 100$?

This can be solved with the bars-and-stars methods using 3 bars and 100 stars, with the numbers of stars to the left of the three bars corresponding to the values of x, y and z. It follows that the answer should be $\binom{100+3}{3} = \binom{103}{3} = 5151$.

(6) How many sets $\{x, y, z\}$ of three nonnegative integers are there with $x < y < z < 100$?

This is a simple problem about combination (which is put here for you to compare it with the previous two problems): to specify such a set is just to pick three distinct numbers from the set $\{0, 1, 2, \ldots, 100\}$. There are $\binom{101}{3}$ ways to do so, so there are $\binom{101}{3}$ sets of the desired form.

(7) How many permutations are there of the letters in the word "PEPPERMINT"?

This is a standard word problem. There are 10 letters in total in the multiset of letters involved in the word, and the multiplicities of the elements are $1, 1, 1, 1, 1, 2$ and 3, so there are $\frac{10!}{2!3!}$ permutations of the word "PEPPERMINT".