Math 2001. Lecture 9.

01. 31. 2022.

Last time: · the multiplication, addition, and subtraction principles for counting

· general strategy: · use mult. principle to simplify things when possible

· use addition principal for cases, making sure not to over count

· recognize when the subtraction principle may be useful

· Overall, do not miss objects, do not overcount.

E.g. Ex 3.2.5: How many integers between 1 and 9999 have at least one repeated digit? (eg 747)

$$9999 - (9 + 9\cdot 9 + 9\cdot 9\cdot 8 + 9\cdot 9\cdot 8\cdot 7)$$

Today: · more practice problems · permutations and combinations

1. Practice problems

Bonus: what about all blacks $\supseteq$ or all spades ?

$26 \cdot 25 \cdot 24 \cdot 23 \cdot 22$

(1) Five cards are taken out of a standard 52-card deck and lined up in a row.

- How many such line ups are there with at least one red card?

$$52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 - (\underbrace{26 \cdot 25 \cdot 24 \cdot 23 \cdot 22}_{\text{all-black}})$$

- _ _ _ _ _ _ _ _ in which the cards are either all blacks or all hearts?

red

$$26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 + 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9$$

- _ _ _ _ _ _ _ of the same color?

$$\underbrace{26 \cdot 25 \cdot 24 \cdot 23 \cdot 22}_{\text{all-red}} + \underbrace{26 \cdot 25 \cdot 24 \cdot 23 \cdot 22}_{\text{all-black}}$$

Ex 3.3.7. A password must be five characters long, made from the English alphabet, and have at least one upper case letter. (1) How many different passwords are there? (2) What if we want a mix of upper and lower cases?

No upper case $\equiv$ all small case : 26^5

Overall (no restriction) : 52^5

(1) $52^5 - 26^5$

(2) $52^5 - (\underbrace{26^5}_{\text{upper only}} + \underbrace{26^5}_{\text{lower only}})$

2. Factorials, permutations, and Combinations

Def: For each $n \in \mathbb{Z}_{\geq 1}$, we define the factorial of n to be the number

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 1$$

e.g. $0! = 1$, $1! = 1$

$2! = 2 \times 1 = 2$,

$3! = 3 \times 2 \times 1 = 6$

$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1$

$= 720$.

We define the factorial of 0 to be $0! = 1$

Def: (Permutations / k-permutations) Let X be a set of n elts. A k-permutation of X is a non-repetitive list of k elts of X. If $k = n$, we also just call a k-permutation a permutation of X. (using all elts of X)

By the last lecture (the mult principle),

$$\#(\text{k-permutations of } X \text{ if } |X| = n) = \underbrace{n \cdot (n-1) \cdots (n-k+1)}_{k \text{ factors}}$$

$$n!/(n-k)! \;=\; n \cdot (n-1) \cdots (n-k+1)$$

Key: for permutation, no rep is allowed, and order matters

Eg (3.10) 5 cards out of 52-card deck, lined up.

How many such lineups have only reds or only clubs? (black)

$$26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 + 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9.$$

Permutations vs. combinations

Permutations should be contrasted with "combinations".

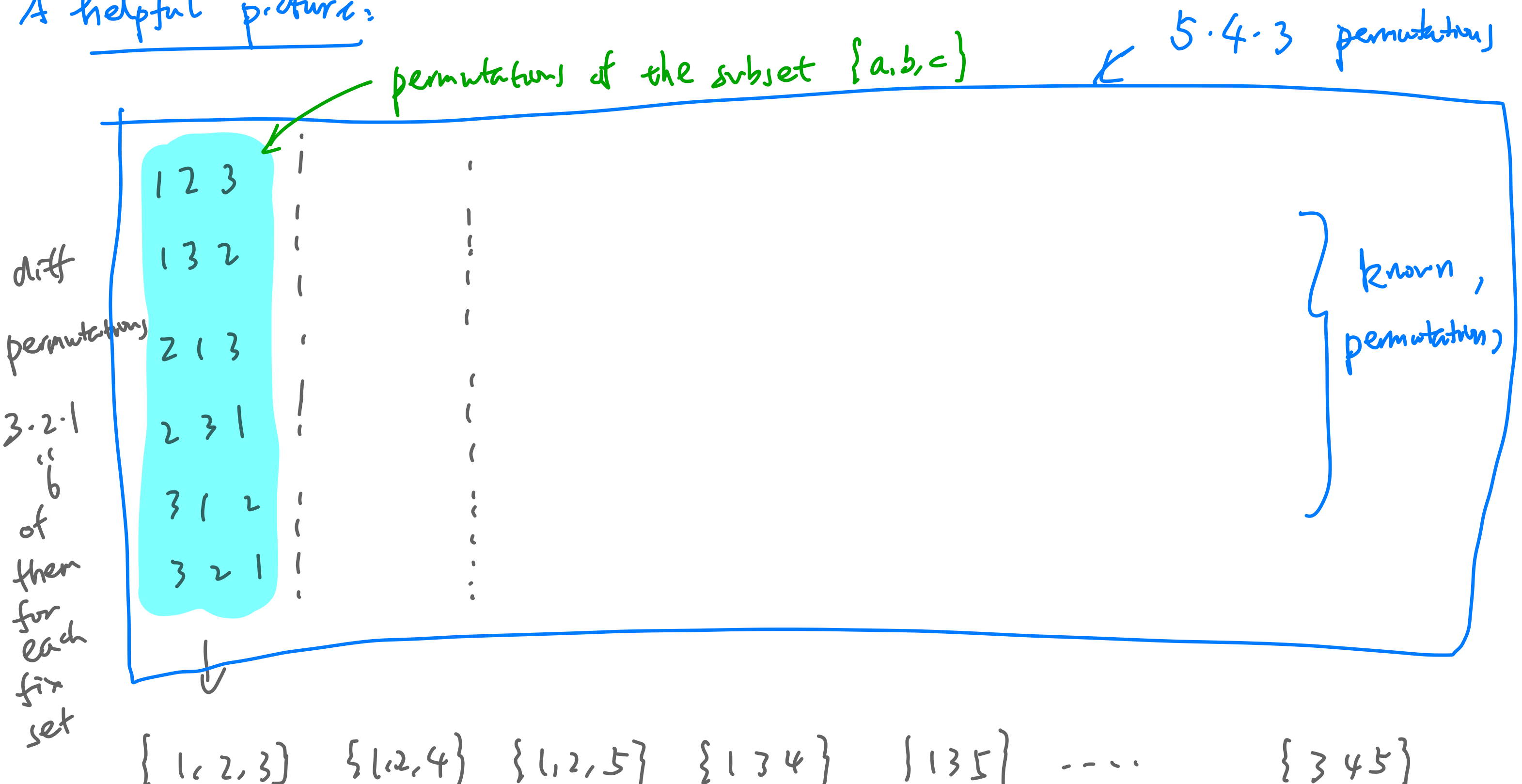
Consider the following two tasks: Let $X = \{a, b, c, d, e\}$, $|X| = 5$.

(1) Permutation: Take 3 elts outs of X and line them up:

$$5 \cdot 4 \cdot 3 = \frac{5!}{2!}$$

(2) Combination/Subset: Take 3 elts of X and form a subsets. How many such subsets are there? Order doesn't matter: $\{1,3,4\} = \{3,4,1\}$

A helpful picture:



Point: If we counted the sets first "as permutations", we'd count each set $3 \cdot 2 \cdot 1 = 6$, so the desired number of sets should be $\dfrac{5 \cdot 4 \cdot 3}{3 \cdot 2 \cdot 1}$.

A smaller, full example:

permutations of the subset $\{a,b\}$

2-permutations of $\{a,b,c,d\}$	ab	ac	ad	bc	bd	cd
	ba	ca	da	cb	db	dc
vs.	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
Subsets of size 2 of $\{a,b,c,d\}$	$\{a,b\}$	$\{a,c\}$	$\{a,d\}$	$\{b,c\}$	$\{b,d\}$	$\{c,d\}$

We have explained how to count subsets/combinations of k objects from X:

Prop: If $|X| = n$, then

$$\#\left(\text{sets/combinations of } k \text{ elts from } X\right) = \frac{\#\, k\text{-permutations of } X}{k!} = \frac{n!}{(n-k)!\,k!}$$