Math 2001. Lecture 8.

01.28.2022.

Last time:

- DeMorgan's Law
- Logic vs. sets
- Open sentences and quantifiers.

Today. start Ch.3. Counting.

— the multiplication, addition and subtraction/complement principles

— Counting problems.

1. The principles

The multiplication principle. Suppose a task T can be completed in n independent steps $S_1, S_2, \cdots, S_n$, and suppose that each step S_i can be done in a_i ways. Then the task can be done in $a_1 a_2 \cdots a_n$ ways.

(what's done in one step doesn't affect things in other steps)

Eg. (3.2) In order a latte (T), you have a choice of whole, skim or soy milk (S_1); small, medium or large (S_2); and either one or two shots of espresso (S_3). How many choices do you have in ordering one drink?

Answer: $\underbrace{3}_{a_1} \cdot \underbrace{3}_{a_2} \cdot \underbrace{2}_{a_3} = 18$. There are 18 choices in total.

Rmks:

- We have in fact used the mult. principle before, in counting Cartesian products of sets and counting the power set $\mathcal{P}(A)$ of a set $A=\{x_1, x_2, \cdots x_i, \cdots\}$

$$2^{|A|} = \underset{a_1}{\underline{2}} \cdot \underset{a_2}{\underline{2}} \cdot \underset{a_3}{\underline{2}} \cdot \cdots \cdot 2 \leftarrow$$

T: Specify a subset

S_i: to include the ith elt or not

$a_i = 2 \; \forall i$.

- Sets offer a nice framework for counting and using the counting principles.

Indeed, the mult. principle is exactly equiv to the principle for counting Cartesian products.

- We also have the addition principle and subtraction principle, phrased in terms of sets as follows:

no two of the subsets intersect.

+: If a set X is the pairwise disjoint union of a number of subsets $X_1, X_2, \cdots, X_n$.

Then $|X| = |X_1| + |X_2| + \cdots |X_n|$.

- : For a set X in a universe U, $|X| = |U| - |\overline{X}|$.

2. More counting problems

(~~Lists~~ Tuples are always ordered in this course.)

(1) Consider lists of length 4 made with symbols A, B, C, D, E, F.

(a) How many such lists are possible if repetition is allowed? (e.g. ABCB ✓, ABBD ✓)

T: specifying the four letters □□□□.

S_i ($i = 1, 2, 3, 4$) : specifying the i th letter. $\rightsquigarrow a_i = 6 \ \forall i$.

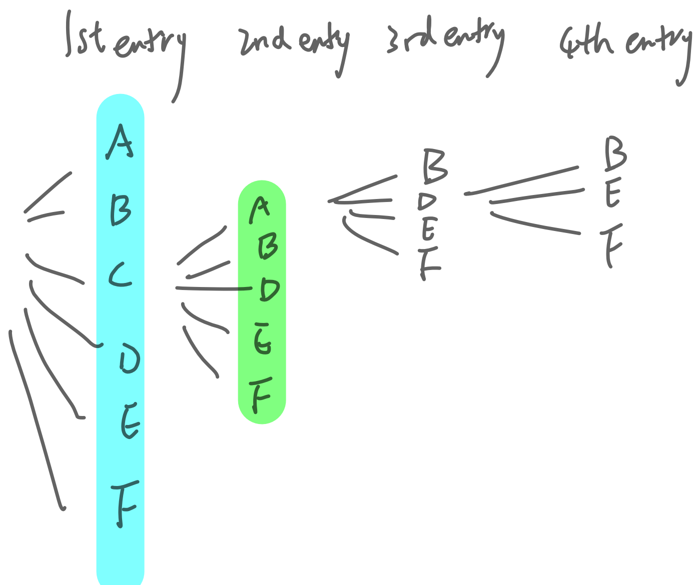
Soln: Each entry in the list can be specified independently in 6 ways,

so there are $6 \cdot 6 \cdot 6 \cdot 6 = 1296$ ways.

(b). How many such lists are possible if repetition is not allowed?

Analysis: T: same, but with repetition.

S_i: choose the entry for the ith step after the previous entries have been chosen. $\rightarrow$ $a_1 = 6$, $a_2 = 5$, $a_3 = 4$, $a_4 = 3$.



Soln: There are 6 options for choosing the first entry and then 5 choices for the second and then 4 choices for the third and so on, so there are $6 \cdot 5 \cdot 4 \cdot 3 = 360$ such lists.

(c) How many such lists are there if repetition is not allowed and the list has an E ?

S_1 : decide where to put the E $\longrightarrow$ $a_1 = 4$

S_2 : fill the three remaining positions with letters from $\{A, B, C, D, F\}$, with no rep. $\longrightarrow$ $a_2 = 5 \cdot 4 \cdot 3$.

Where for E ?

fill the empty positions, old problem

$E \_ \_ \_$

$\_ E \_ \_$ $\longrightarrow$ $5 \cdot 4 \cdot 3$

$\_ \_ E \_$

$\_ \_ \_ E$

Soln: There 4 choices for deciding where to put E and then $5 \cdot 4 \cdot 3$ choices for filling the rest of the list, so there are $4 \cdot (5 \cdot 4 \cdot 3) = 240$ such lists.

(d) How many such lists are there if rep. is allowed and the list has an E?

Suggestion 1: $4 \cdot 6^3$ how to fill the rest with $\{A, \cdots, E, F\}$

where to put an E

too much, problem: E E A B gets double-counted as E (E A B)

and E E A B

Soln: We use the subtraction principle: There are 5^4 lists of length 4 w/ repetition allowed and no E at all, the total number of lists of length w/ rep. allowed is 6^4.

So there are $6^4 - 5^4$ lists of length 4, rep. allowed, that have an E.

12). Make a non-repetitive list of length 5 from the symbols A, B, C, D, E, F, G.

The first entry must be B, C, or D, and the last entry must be a vowel.

How many such lists are there?

Soln:

$$\underbrace{3}_{\text{first entry } B,C, \text{ or } D} \cdot \underbrace{2}_{\text{last entry, } A \text{ or } E} \cdot \underbrace{5 \cdot 4 \cdot 3}_{\text{5 letters left for the remaining positions.}} = 360.$$

Next time:

- more counting problems
- permutations and combinations

B _ _ _ _ B _ _ _ E {A, C, D, F, G}

_ _ _

Q: What if we ask the first entry be from A, C, D instead?

EX: try this with the addition principle (break thing down to disjoint cases first).