Math 2001. Lecture 7.

01.26.2022.

Last time:

- Conditional statements
- how to prove different types of statements
- logical equivalences

Today:

- One more logical equivalence: DeMorgan's Law
- from logic to sets: 'and' $\leftrightarrow$ $\cap$, 'or' $\leftrightarrow$ $\cup$.

  'not' $\leftrightarrow$ $\overline{}$ (complement)
- open sentences and quantifiers

1. De Morgan's Law

Prop 1: Let P, Q be statements. Then we have the following equivalences ($\equiv$)

① $\sim(P \wedge Q) \equiv ((\sim P) \vee (\sim Q))$ (compare with $\overline{A \cap B} = \bar{A} \cup \bar{B}$)

② $\sim(P \vee Q) \equiv ((\sim P) \wedge (\sim Q))$. (. . . . . . $\overline{A \cup B} = \bar{A} \cap \bar{B}$)

We'll check ① with a truth table:

P	Q	$P \wedge Q$	LHS	$\sim P$	$\sim Q$	RHS
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

✓

2. From logic to sets

By their definitions, taking the union, intersection, and complement of sets naturally correspond to the 'or', 'and', & 'not' operators in logic...

E.g. Proving the set version of DeMorgan's Law using the logic version:

(Imagine a universe U.) Let $A \subseteq U$, $B \subseteq U$ be two sets (in the universe).

For all $x \in U$, let P be the statement "$x \in A$" and Q be the statement "$x \in B$". Then $P \wedge Q$ is the statement "$x \in A \cap B$" by the def. of $\cap$.

$P \vee Q$ — · — ·· "$x \in A \cup B$" ··· ··· of $\cup$.

By Prop 1. ①.

$$\sim(\underbrace{P \wedge Q}_{x \in A \cap B}) = (\sim P) \vee (\sim Q),$$

so

$$(x \notin A \cap B) = ((x \notin A) \vee (x \notin B))$$

i.e.

$$(x \in \overline{A \cap B}) = ((x \in \bar{A}) \vee (x \in \bar{B}))$$

def of $\cup$

i.e.

$$(x \in \overline{A \cap B}) = (x \in \bar{A} \cup \bar{B})$$

Since the above is true for all x, it follows that $\overline{A \cap B} = \bar{A} \cup \bar{B}$.

The fact that $\overline{A \cup B} = \bar{A} \cap \bar{B}$ can be proved similarly $\square$

3. Open sentences and quantifiers

E.g. "x is an odd integer" is not a statement; it's an open sentence in that we can only judge its truth after x is made more specific.

We need quantifiers for x to make it a statement.

There are two main quantifiers in math:

- the universal quantifier "for all", written "$\forall$" → "$\forall x \in \mathbb{Z}$, x is an odd int." → False
- the existential quantifier "for some", written "$\exists$" → "$\exists x \in \mathbb{Z}$, x is an odd int." → True

Note:

- make sure you use quantifiers when necessary!
- ("any") Avoid "any", use "every / each" or "some" to avoid ambiguity.
- "All that glitters is not gold." ⟶ clearer: "Not all that glitters is gold."

Eg. Write the following in English. Determine whether they are true or false.

(1) $\forall x \in \mathbb{R},\ x^2 > 0$. $\longrightarrow$ false. $x=0$ is a counter-example.

"For every real number x, we have x^2 is positive".

"Every real number squares to a positive number".

(2) $\exists a \in \mathbb{R}$ s.t. $\forall x \in \mathbb{R},\ ax = x$ $\longrightarrow$ True, $a=1$ works.

"There is a real number a such that $ax = x$ for every real number x".

(3) $\forall n \in \mathbb{Z},\ \exists m \in \mathbb{Z}$ s.t. $m = n+5$. $\longrightarrow$ True.

"For every integer n, there is another integer m satisfying $m=n+5$."

(4). $\exists m \in \mathbb{Z}$ s.t. $\forall n \in \mathbb{Z},\ m = n+5$. $\longrightarrow$ False.

"There is an integer m s.t. for all integer n we have $m=n+5$."

The order of the quatifiers matters.