Math 2001. Lecture 6.

01. 24. 2022.

Last time:

- indexed sets
- statements ; and, or, not ; truth tables

Today:

- conditional statements
- how to prove various types of statements
- logical equivalences

1. Conditional statements

By a conditional statement we mean a statement that can be made in the form "if P, then Q" where P, Q are themselves some statements.

Running example: if $\underbrace{x \text{ is an integer divisible by } 6}_{P}$, then $\underline{x \text{ is an even integer}}$.

Note: A conditional statement can often be stated in various ways:

(P44)

if P, then Q; P is sufficient / a sufficient condition for Q;

Q if P; Q is necessary / a necessary condition for P;

Q whenever P; P only if Q;

Q, provided that P;

Whenever P, Q;

notation for "if P then Q": $P \Rightarrow Q$.

To prove "if P then Q", assume P is true and derive Q.

E.g. Prop: If $\underbrace{x \text{ is an int. divisible by } 6}_{P}$, then $\underbrace{x \text{ is an even int.}}_{Q}$

Pf: Suppose x is an int. divisible by 6.

Note the template!

Then $x = 6k$ for some int. k.

It follows that $x = 6 \cdot k = (2 \cdot 3) \cdot k = 2 \cdot (3k)$,

where $3k$ is an integer.

So x is an even integer.

We have proved the proposition. $\square$

Note: · To prove "if P then Q", we don't need to worry about the cases where P is false (eg $x=5$); in those cases, the statement "if P then Q" is "vacuously true".

· We'll often see statements "$P \Leftrightarrow Q$", which means "$P \Rightarrow Q$ and $Q \Rightarrow P$" and are often phrased as "P if and only if Q".
"iff"

the arrow always points from the hypothesis to the consequence

· We define the converse of "if P then Q" to be the statement "if Q then P". The converse of a true statement may not be true, and vice versa.

Eg P: x is an int. divisible by 6; Q: x is an even integer. ⟶ $P \Rightarrow Q$: true, as just proved. $Q \Rightarrow P$: false, eg. $x=4$.

2. Summary: proofs of statements

To prove $P \wedge Q$, prove both P and Q.

$P \vee Q$, prove one of P or Q.

$\sim P$, prove P is false.

$P \Rightarrow Q$, assume P and derive Q.

$P \Leftrightarrow Q$, prove $P \Rightarrow Q$ and $Q \Rightarrow P$ separately.

3. Logical equivalences

We say two statements are (logically) equivalent if they always have the same truth value (no matter what the truth values of their component statements are).

Non-example: $P \Rightarrow Q$ and $Q \Rightarrow P$ for the running example.

	P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$
e.g. $(x = 4) \rightarrow$	F	T	T (vacuously true)	F

Conclusion:

"$P \Rightarrow Q$" is not equivalent to "$Q \Rightarrow P$" here.

Example: "$P \Leftrightarrow Q$" is equivalent to "$\underbrace{\big((P \wedge Q) \vee ((\sim P) \wedge (\sim Q))\big)}_{R}$"

for all statement P, Q "both or neither"

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$P \Leftrightarrow Q$	$P \wedge Q$	$\sim P$	$\sim Q$	$(\sim P) \wedge (\sim Q)$	R
T	T	T	T	T	T	F	F	F	T
T	F	F	T	F	F	F	T	F	F
F	T	T	F	F	F	T	F	F	F
F	F	T	T	T	F	T	T	T	T

same!

Example. Every conditional statement "$P \Rightarrow Q$" is equivalent to its "contrapositive" statement, the statement that "$\sim Q \Rightarrow \sim P$".

P	Q	$P \Rightarrow Q$	$\sim Q$	$\sim P$	$\sim Q \Rightarrow \sim P$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

same!

Next time:

- open sentences, quantifiers
- some concrete proofs.

Ex: Show that for all statements P, Q, "$P \vee Q$" is equiv. to "$(\sim P \Rightarrow Q) \wedge (\sim Q \Rightarrow P)$".