Math 2001. Lecture 5.

01. 21. 2022.

Last time:

- using Venn diagrams
- DeMorgan's Laws: $\overline{A \cap B} = \bar{A} \cup \bar{B}$, $\overline{A \cup B} = \bar{A} \cap \bar{B}$
- Proofs of set equalities

Today:

- index sets (notation)
- Statements and combinations of them
- truth table

1. Indexed sets (§ 1.8)

Motivation: Sometimes we deal with a large collection of sets at a time.

We need efficient notation.

Eg. · If we want to consider a collection of sets $A_1, A_2, A_3, \ldots, A_9, A_{10}, \ldots$

We may write $\bigcup_{i=1}^{9} A_i$ as a shorthand for the union

$$A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6 \cup A_7 \cup A_8 \cup A_9$$

We may also write $\bigcap_{i=1}^{\infty} A_i$ for the intersection

$$A_1 \cap A_2 \cap A_3 \cap A_4 \cap \cdots\cdots$$

Often the set indices form a set I themselves

So it's natural to define

See E.g. 1.14 in the book, where $I = [0, 2]$.

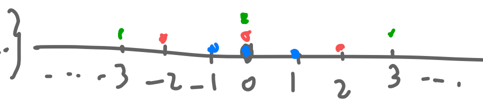
$$\bigcup_{i \in I} A_i = \{ x \mid x \text{ is in } A_i \text{ for at least one index } i \in I \}$$

$$\bigcap_{i \in I} A_i = \{ x \mid x \text{ is in } A_i \text{ for every index } i \in I \}$$

Examples:

· If $A_1 = \{-1, 0, 1\}$, $A_2 = \{-2, 0, 2\}$, $A_3 = \{-3, 0, 3\}$, ...

Then $\bigcup_{i \in I} A_i = \{ \cdots, -3, -2, -1, 0, 1, 2, 3, \cdots \}$



$\bigcap_{i \in I} A_i = \{ 0 \}$

$\bigcup_{i=3}^{6} A_i = \{-6, -5, -4, -3, 0, 3, 4, 5, 6\}$.

- If $A_n = \{0, 1, 2, 3, \cdots, n\}$ for each $n \in \mathbb{Z}_{\geq 0}$, then

$$\bigcup_{i=0}^{\infty} A_i = \mathbb{Z}_{\geq 0}$$

$$\bigcap_{i=0}^{\infty} A_i = A_0 \cap A_1 \cap A_2 \cap \cdots = \{0\} \cap \{0, 1\} \cap \{0, 1, 2\} \cap \cdots = \{0\}.$$

- For any collection of sets A_α with index set I, we have

$$\bigcap_{i \in I} A_i \subseteq \bigcup_{i \in I} A_i \quad ;$$

in fact, $\bigcap_{i \in I} A_i \subseteq A_j \subseteq \bigcup_{i \in I} A_i \quad \forall j \in I$.

Q: Can you prove these two set containments?

2. Statements (§2.1)

A statement is a sentence or mathematical assertion that is either definitely true or definitely false.

Examples and non-examples

- $2 \in \mathbb{Z}$ (The number 2 is an integer): a statement, a true one.
- $\pi \in \mathbb{Z}$: a statement, false
- $\mathbb{Z} \subseteq \mathbb{R}$: a statement, true.
- 42 : not a statement, no assertion is made

- "What is the soln of $2x = 84$?" Not a statement → it's a question

- Add 3 to 5. Not a statement → It's an instruction

- "If n is an even integer, then $n+1$ is an odd integer."

  This is a statement.

  It's an "if--then statement", and it's true.

- "If n is an odd integer, then $2n$ is an odd integer".

  a false statement.

- $\underbrace{2 \in \mathbb{Z} \text{ and } \pi \in \mathbb{Z};\ 2 \in \mathbb{Z} \text{ or } \pi \in \mathbb{Z};\ \text{it's not the case that } \pi \notin \mathbb{Z}}_{\text{three statements}}$.

3. New statements from old ones: and, or, not

We can often produce more complex statement out of simpler ones.

We are interested in how the validity of the simpler statement affect the validity of the new statement

Notation:

- "T": True, "F": False.
- We'll often denote statements by letters such as $P, Q, R, S, \dots$
- We write "$\wedge$" for "and", "$\vee$" for "or", and "$\sim$" for "not".

Eg. If P is the statement "$2 \in \mathbb{Z}$" (T)

and Q is the statement "$\pi \in \mathbb{Z}$" (F).

Then $P \wedge Q$ stands for the statement "$2 \in \mathbb{Z}$, and $\pi \in \mathbb{Z}$" (F)

$P \vee (\sim Q)$ - - - - - - - - "$2 \in \mathbb{Z}$ or $\pi \notin \mathbb{Z}$" (T).

Note that we have the following truth tables:

"and"	P	Q	$P \wedge Q$
	T	T	T
	T	F	F
	F	T	F
	F	F	F

"or"	P	Q	$P \vee Q$
	T	T	T
	T	F	T
	F	T	T
	F	F	F

"not"	P	$\sim P$
	T	F
	F	T

Eg. Suppose we have

P	Q	R	S	O
T	F	T	F	F

.

Then we can decide if $P \vee Q \vee (R \wedge (S \vee (\sim O)))$ is T/F.

$$\underbrace{\underset{T}{P} \vee \underset{F}{Q} \vee \underbrace{(\underset{T}{R} \wedge \underbrace{(S \vee \underbrace{(\sim O)}_{T})}_{T})}_{T}}_{T\,!}$$

Next time: · conditional statement

"if P then Q"