Math 2001. Lecture 4.

01.19.2022.

Last time:

- Counting power sets $|P(A)| = 2^{|A|}$
- Set operations: unions, intersections, complements
- Venn diagrams

Today:

- more visualizations of sets using Venn diagrams.
- more proofs of set equalities, DeMorgan's Laws

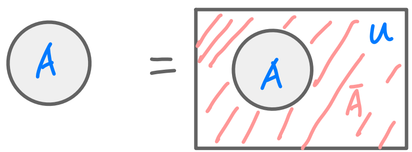
Principle:

- To prove $A = B$ for two sets A, B, prove $A \subseteq B$ and $B \subseteq A$.
- To prove $X \subseteq Y$, show that every elt $x \in X$ is also in Y.

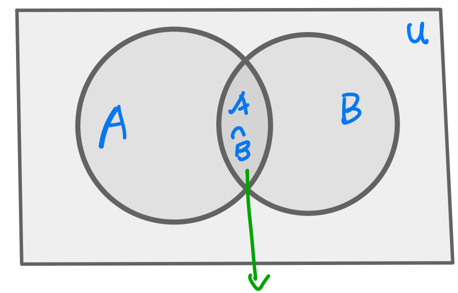
1. Venn diagrams

How we typically depict the interactions of one, two, three, four sets (in a universe)

1:

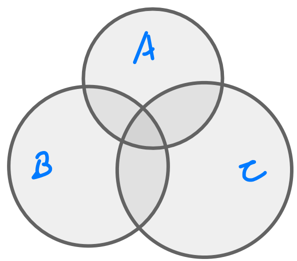


2:

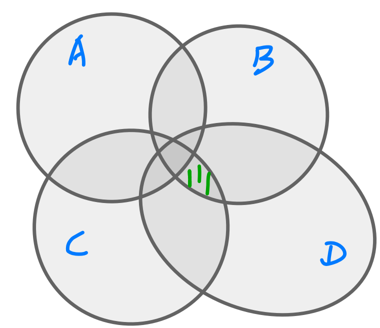


$A \cap B$ is possibly empty

3:



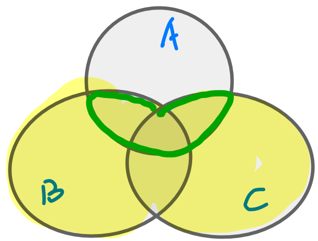
4.

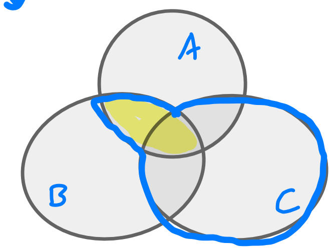


/// = $B \cap C \cap D$.

Advantage: help visualize the results of different set operations as regions in the pictures.

E.g. LHS $A \cap (B \cup C) \stackrel{?}{=}$ RHS $(A \cap B) \cup C$ (Are the two sets always equal for three sets A, B, C?)





By the Venn diagrams, we expect the two sets to be different.

the problem

Moreover, any elt in $C - (A \cup B)$, if such an elt exists, is in the right set but not in the left set.

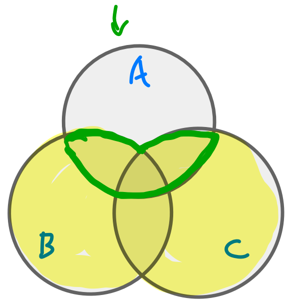
This leads to a counter-example.

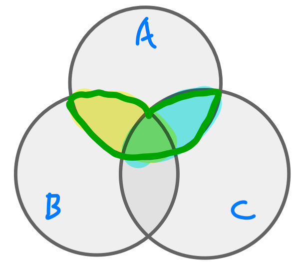
It follows that "RHS" $\nsubseteq$ "LHS", so $(A\cap B) C \nsubseteq A \cap (B \cup C)$.

Answer: No, $A \cap (B \cup C) \neq (A \cap B) \cup C$. For example, take $A = \{1, 2\}$, $B = \{2, 3\}$, $C = \{3, 4, 5, 6\}$. Then $6 \in (A \cap B) \cup C$ since $6 \in C$, but $6 \notin A \cap (B \cup C)$ since $6 \notin A$.

E.g.

$$A \cap (B \cup C) \stackrel{?}{=} (A \cap B) \cup (A \cap C)$$





Q: We seem to have equality here. How do we prove it?

$\longrightarrow$ next section.

2. Proofs of set equalities

Principle: The most common way to prove $A=B$ for two sets is to prove both $A \subseteq B$ and $B \subseteq A$.

Eg. Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

By the above parts, it follows that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Pf: (LHS $\subseteq$ RHS): Let $x \in A \cap (B \cup C)$. Then $x \in A$, and $x \in B \cup C$, hence x is in B or in C. If $x \in B$, then $x \in A \cap B$ and hence $x \in (A \cap B) \cup (A \cap C)$; if $x \in C$, then $x \in A \cap C$ and hence $x \in (A \cap B) \cup (A \cap C)$. Either way, $x \in (A \cap B) \cup (A \cap C)$. Therefore $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

(RHS $\subseteq$ LHS). Let $x \in (A \cap B) \cup (A \cap C)$. Then $x \in A \cap B$ or $x \in A \cap C$. If $x \in A \cap B$, then $x \in A$, and $x \in B \cup C$ since $x \in B$, so $x \in A \cap (B \cup C)$. If $x \in A \cap C$, then $x \in A$, and $x \in B \cup C$ since $x \in C$, so $x \in A \cap (B \cup C)$. Either way, $x \in A \cap (B \cup C)$. Therefore $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.

Eg. Prop. (DeMorgan's Laws) Let A, B be two sets in a universe $\mathcal{U}$.

Then (i). $\overline{A \cup B} = \bar{A} \cap \bar{B}$

(ii). $\overline{A \cap B} = \bar{A} \cup \bar{B}$.

→ Ex: visualize this with Venn diagrams.

Pf: We sketch the proof of (i) and leave (ii) as an exercise.

(i). ($LHS \subseteq RHS$) Let $x \in \overline{A \cup B}$. Then x is not in $A \cup B$.

So x is not A and is not in B, so $x \in \bar{A}$ and $x \in \bar{B}$.

Therefore $x \in \bar{A} \cap \bar{B}$. Therefore $LHS \subseteq RHS$.

($RHS \subseteq LHS$) Let $x \in \bar{A} \cap \bar{B}$. Then $x \in \bar{A}$ and $x \in \bar{B}$. So x is neither in A and nor in B. So x is not in $A \cup B$, i.e., $x \in \overline{A \cup B}$.

Therefore $RHS = LHS$.

By the above, $\overline{A \cup B} = \bar{A} \cap \bar{B}$.

Ex. Prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Next time:

indexed sets,

logic (Ch. 2.)