Math 2001. Lecture 37. 04. 25. 2022.

Last time: · partitions from equiv. rels: If R is an equiv. rel. on a set A, then

· $[a] = [b] \iff aRb$

· $\{[a] : a \in A\}$ partition A.

Today: · (familiar) basic notions on functions

domain, codomain, composition, injectivity, surjectivity.

Next time: Course review

1. Definitions and notations

Def (Def 12.1 , function) A function from a set A to a set B is a rule that assigns one (unique) elt in B to each elt in A. If an elt $a \in A$ is assigned the output $b \in B$, we write $f(a) = b$.

another way to encode this: (a, b)

Def (domain and range) For a function $f : A \to B$, we call A the <u>domain</u> of f and B the <u>codomain</u> of f.

The <u>range</u> or <u>image</u> of f is the set $\mathrm{im}(f) = \{ f(a) : a \in A \} \subseteq B$.

<u>Note:</u> In general the image of a function f may not equal its codomain.

E.g. For $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$, we have $\mathrm{codomain}(f) = \mathbb{R}$ while $\mathrm{im}(f) = \{ x^2 : x \in \mathbb{R} \} = \{ y : y \geq 0 \} = \mathbb{R}_{\geq 0}$.

Def (Def 12.3, equality of functions) Two functions $f: A \to B$ and $g: C \to D$ are equal if $\underbrace{A = C}_{\text{same domain}}$ and $\underbrace{f(a) = g(a) \quad \forall a \in A = C}_{\text{same behavior on the entired shared domain}}$.

E.g. $$\left(f: \mathbb{R} \to \mathbb{R},\ f(x) = x^2 \ \forall x \in \mathbb{R} \right) = \left(g(x): \mathbb{R} \to \mathbb{R}_{\geq 0},\ g(x) = x^2 \ \forall x \in \mathbb{R}. \right)$$

Def (Def 12.5. composition) Suppose $f: A \to B$ and $g: B \to C$ are functions. Then the $\underline{\text{composition of } f \text{ with } g}$ is the function $g \circ f: A \to C$ s.t.

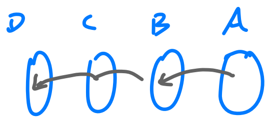
$$g \circ f\,(a) = g(\underbrace{f(a)}_{B})) \qquad \forall a$$

Eg if $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ are given by $f(x) = x+1$ and $g(x) = x^2$ $\forall x \in X$, then
$(g \circ f)(x) = g(f(x)) = g(x+1) = (x+1)^2 = x^2 + 2x + 1$.

Thm. (Thm 12.5) Composition is associative, that is, if $f: A \to B$, $g: B \to C$, and $h: C \to D$ are functions, then $(h \circ g) \circ f = h \circ (g \circ f)$.

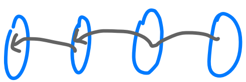
$D \leftarrow B \quad B \leftarrow A \qquad C \leftarrow A$

Pf: Both $(h \circ g) \circ f$ and $h \circ (g \circ f)$ have domain A, and $\forall a \in A$.



$$[(h \circ g) \circ f](a) = (h \circ g)(f(a)) = h(g(f(a)))$$

$$\parallel$$



$$[h \circ (g \circ f)](a) = h((g \circ f)(a)) = h(g(f(a)))$$

So $[(h \circ g) \circ f](a) = [h \circ (g \circ f)](a)$, therefore $(h \circ g) \circ f = h \circ (g \circ f)$.

2. Injectivity, Surjectivity, and bijectivity

Contrapositives

Def (Def 12.4) A function $f: A \to B$ is

(1) injective (or one-to-one) if $\forall a, a' \in A$, we have $f(a) = f(a') \Rightarrow a = a'$.

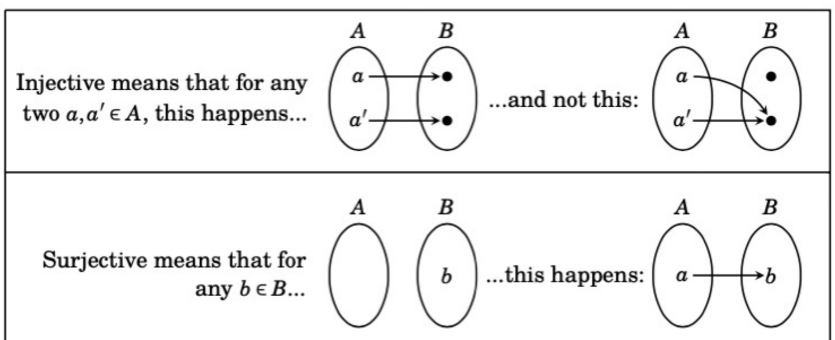
equivalently, if $\forall a, a' \in A$, we have $a \neq a' \Rightarrow f(a) \neq f(a')$

(2) surjective (or onto) if $\forall b \in B$, $\exists a \in A$ s.t. $f(a) = b$.

equivalently, if $B = \mathrm{Im}(f)$.

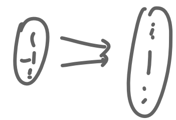
(3) bijective if it's both injective and surjective.

In pictures:



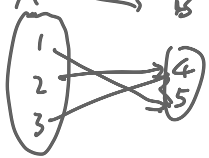
First Examples:

- The function $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2 \ \forall x \in \mathbb{R}$ is neither inj nor. surj:

  $1 \neq -1$, but $f(1) = f(-1)$,  so f is not inj;

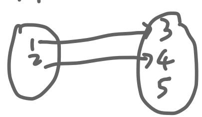
  $f(x) = x^2 \geq 0$, so $-1 \in \mathbb{R}$ is not in $\mathrm{im}(f)$, so f is not surj.

- $f: A \to B$

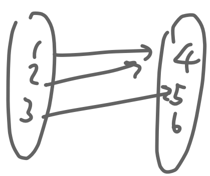
  

  is surj. but not inj ($2 \neq 3$ but $f(2) = f(3)$).

- $f: A \to B$ is inj but not surj.

- $f: A \to B$

  is neither surj. nor inj.

How to prove a function $f: A \to B$ is ...

(1) inj: suppose $f(a) = f(a')$ for some $a, a' \in A$ and deduce $a = a'$.

(2) surj: let $b \in B$ and argue that $b = f(a)$ for some $a \in A$.

(3) bij: prove f is inj and surj.

$$\boxed{A \setminus B = \{a \in A : a \in B\}}$$

Examples (sketch)

12.4. $f: \mathbb{R} \setminus \{0\} \to \mathbb{R}$ w/ $f(x) = \frac{1}{x} + 1 \quad \forall x \in \mathbb{R} \setminus \{0\}$.

surj? No, 1 is in the codomain $\mathbb{R}$, but $1 \neq \frac{1}{x} + 1 \quad \forall x \in \mathbb{R} \setminus \{0\}$.

inj? Yes: $\frac{1}{x_1} + 1 = \frac{1}{x_2} + 1 \Rightarrow \frac{1}{x_1} = \frac{1}{x_2} \Rightarrow x_1 = x_2 \quad \forall x_1, x_2$.

12.2.5. $f: \mathbb{Z} \to \mathbb{Z}$ w/ $f(n) = 2n + 1 \quad \forall n \in \mathbb{Z}$

surj? No, $2 \neq 2n+1 \quad \forall n \in \mathbb{Z}$.

inj? Yes: $2n + 1 = 2n' + 1 \Rightarrow 2n = 2n' \Rightarrow n = n'$.