Math 2001. Lecture 36.

04. 22. 2022.

Last time:

- verification of equivalence relations
  - congruence relations on $\mathbb{Z}$.
  - "equality of fractions / rational numbers"
- HW.

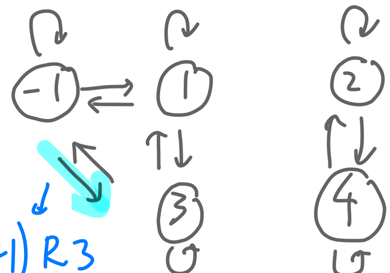
Today:

- partitions from equivalence classes
  - def. of partitions, examples via equivalence classes
  - why we get partitions from equivalence relations

1. Partitions from equivalence classes

Example from earlier: Consider the equivalence relation on $A = \{-1, 1, 3, 2, 4\}$

depicted by



$(-1) R 3$

$\left(aRb \iff \text{"} a \to b \text{"} \right)$.

$\{x \in A : \text{"} x \to a \text{"}\}$

$\parallel$

Recall that $\forall a \in A$, the equivalence class of a is $[a] := \{x \in A : xRa\} \subseteq A$.

Direct computation shows that

$[-1] = \{1, 3, -1\}$ $\quad$ $[1] = \{-1, 3, 1\}$ $\quad$ $[3] = \{-1, 1, 3\}$

$\nearrow$ $[-1] = [1] = [3] = \{-1, 1, 3\}$

$[2] = [4] = \{2, 4\}$

$[2] = \{2, 4\}$ $\quad$ $[4] = \{2, 4\}$.

What's special? The five elts give rise to 2 nonintersecting sets whose disjoint union is A.

To be precise:

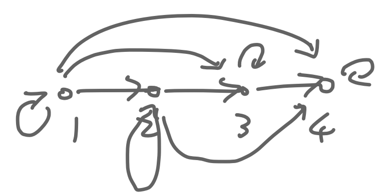
Def (Partition) A partition of a set A is a set of nonempty subsets of A s.t. (1) no two of them intersect nontrivially and (2) their union equals A.

E.g. For $A = \{1, 2, 3, 4, 5\}$.

The sets $\{1, 5\}$, $\{2, 4\}$, $\{3\}$ partition A, as do the set $\{1, 3, 5\}$ and $\{2, 4\}$, as do $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}$.

Note: Given a general relation R on a set A, the distinct sets among the sets $[a] := \{x \in A : xRa\}$ do not necessarily form a partition of A.

Example. $A = \{1, 2, 3, 4\}$, $R =$ "$\leq$"



$\Rightarrow [1] = \{1\}$, $[2] = \{1, 2\}$, $[3] = \{1, 2, 3\}$, $[4] = \{1, 2, 3, 4\}$. $\Rightarrow$ $[1], [2], [3], [4]$ do not partition A.

However, if R is an equivalence rel. on A, then ...

Thm (Thm 11.2) Suppose R is an equiv. rel. on a set A. Then the sets $\{[a] : a \in A\}$ of equiv. classes forms a partition of A.

Examples

- In the fractions example $A = \left\{ \frac{m}{n} \;\middle|\; m, n \in \mathbb{Z},\ n \neq 0 \right\}$ from last time, the equivalence class of $\frac{1}{2}$ is $\left[\frac{1}{2}\right] = \left\{ \frac{m}{n} : m \cdot 2 = n \cdot 1,\ m, n \in \mathbb{Z},\ n \neq 0 \right\}$

  $= \left\{ \frac{m}{2m} : 2m \neq 0 \right\} = \underbrace{\left\{ \frac{m}{2m} : m \in \mathbb{Z},\ m \neq 0 \right\}}_{\updownarrow}$

  In general, the equivalence classes correspond bijectively to rational numbers. (all fraction expressions of) the rational number $\frac{1}{2}$

- $A = \mathbb{Q}$, $R =$ equality.

- (modular arithmetic) If $A = \mathbb{Z}$ and xRy if $x \equiv y \pmod 3$ $\forall x, y \in A$, then R partitions A into three equivalence classes. They are

$[0] = [3] = [-3] = \cdots \quad \cdots = \cdots = \{\ldots, -9, -6, -3, 0, 3, 6, 9, \ldots\}$

$[1] = [4] = [-2] = \cdots \quad \cdots = \{\ldots, -8, -5, -2, 1, 4, 7, 10, \ldots\}$

$[2] = [5] = [-1] = \cdots \quad \cdots = \{\ldots, -7, -4, -1, 2, 5, 8, 11, \ldots\}$

The equivalence classes correspond to remainders modulo 3.

2. Theory — Proof of Thm 1.

We'll first prove the following:

Thm 0 (Thm 11.1) Suppose R is an equiv. rel. on A. Let $a, b \in A$.

Then $[a] = [b]$ iff aRb.

Pf of Thm 1: $([a]=[b] \Leftrightarrow aRb)$

$(\Rightarrow)$. Suppose $[a]=[b]$. Since R is reflexive, we have aRa, hence $a\in[a]$. It follows that $a\in[b]$. So aRb.

$(\Leftarrow)$ Suppose aRb. We show $[a]=[b]$ by showing that $[a]\subseteq[b]$ and $[b]\subseteq[a]$.

$([a]\subseteq[b])$. Take any $c\in[a]$. Then cRa. Since aRb and R is transitive, it follows that cRb, so $c\in[b]$. It follows that $[a]\subseteq[b]$.

$([b]\subseteq[a])$. Since aRb and R is symm. we have bRa. Thus, a similar argument to the above implies that $[b]\subseteq[a]$.

It follows that $[a]=[b]$ whenever aRb.

We have completed the proof. $\square$

Pf of Thm 11.2. (equivalence classes on a set A partition A)

Let R be an equiv. rel. on A. Let $\mathcal{E} = \{ [a] : a \in A \}$.

We show that (the sets in) $\mathcal{E}$ forms a partition of A by showing that

(1) Any two different sets in $\mathcal{E}$, say $[a]$ and $[b]$ w/ $[a] \neq [b]$, have trivial intersection: otherwise $[a] \cap [b]$ contains some elt $c \in A$. We'd then have cRa and cRb, hence aRc (symm.) and cRb, hence aRb (tran.). But then $[a] = [b]$ by Thm 11.1, contradiction.

(2) Every elt $a \in A$ appears in some set in $\mathcal{E}$; this is easy, since $a \in [a] \in$ by reflexivity of R.

It follows that the thm. holds. $\square$