Math 2001. Lecture 36.

04.19.2022.

Last time:

- relations: def and notation (a relation on a set $A \leftrightarrow$ a subset $R \subseteq A \times A$)
- def of equivalence relations: relations that are reflexive, symmetric, and transitive.

Today: More on equivalence relations:

- Verifying a relation is an equivalence relation
- partitions from equivalence classes.

1. Verifying a relation is an equivalence relation

E.g. 1. (Congruence, e.g. 11.13) Let $A = \mathbb{Z}$ and let n be an arbitrary pos. int.
Define xRy for $x, y \in A$ if $x \equiv y \pmod{n}$.

Claim: R is an equivalence relation.

E.g. if $n = 5$, then $27 R 12$ since $27 \equiv 12 \pmod{5}$, and similarly $18 R 3$.

Pf of the claim: We check that R is reflexive, symm, and transitive.

Ref: We need to show xRx $\forall x \in \mathbb{Z}$, i.e., that $x \equiv x \pmod{n}$ $\forall x \in \mathbb{R}$,
This is true since $n \mid 0 = x - x$ $\forall x \in \mathbb{Z}$.

Symm: We need to show that if xRy for some $x, y \in \mathbb{Z}$, then yRx. So suppose
xRy. then $x \equiv y \pmod{n}$, ie, $n \mid x-y$. But then $n \mid -(x-y) = y-x$, therefore $y \equiv x \pmod{n}$
and yRx.

Tran. We need to show that if xRy and yRz for some $x, y, z \in \mathbb{Z}$, then xRz.

Suppose xRy and yRz. Then $n \mid x-y$ and $n \mid y-z$. So

$n \mid (x-y)+(y-z) = x-z$, therefore $x \equiv z \pmod{n}$, i.e., xRz.

It follows that R is an equivalence relation.

E.g. 2. (fractions, P. 213) Let $A = \left\{ \frac{m}{n} \;\middle|\; m, n \in \mathbb{Z}, n \neq 0 \right\}$. Consider the relation R on A defined by "$\frac{a}{b} R \frac{c}{d}$ if $ad = bc$ $\forall \frac{a}{b}, \frac{c}{d} \in A$".

Prove that R is an equivalence rel.

E.g. $\frac{2}{5} R \frac{6}{15}$ since $2 \cdot 15 = 5 \cdot 6$, $\frac{1}{3} R \frac{7}{21}$ since $1 \cdot 21 = 3 \cdot 7$.

$\frac{0}{4} R \frac{0}{17}$ since $0 \cdot 17 = 0 \cdot 4$. ("R is just 'being equal'")

Pf: We check that R is reflexive, symm. and transitive.

(1). Take $x \in A$. Then $x = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$, $b \neq 0$.

We need to show that xRx, i.e., $\frac{a}{b} R \frac{a}{b}$. This holds since $ab = ab$.

2) Suppose xRy for some $x, y \in A$, say $x = \frac{a}{b}$ and $y = \frac{c}{d}$ for some $a, b, c, d \in \mathbb{Z}$ $(b \neq 0, d \neq 0)$. Since $\frac{a}{b} R \frac{c}{d}$, we have $ad \overset{(1)}{=} bc$.

To prove yRx, we need $\frac{c}{d} R \frac{a}{b}$, i.e. we need $cb \overset{(2)}{=} da$, which holds by (1), so yRx.

(3). Suppose xRy and yRz for some $x, y, z \in A$, say w/ $x = \frac{a}{b}$, $y = \frac{c}{d}$, $z = \frac{e}{f}$ for some $a, b, c, d, e, f \in \mathbb{Z}$ with b, d, f nonzero. Since xRy, we have $ad \overset{(3)}{=} bc$; since yRz, we have $cf \overset{(4)}{=} de$. Equations (3), (4) imply $adcf \overset{(5)}{=} bcde$.

We need to show that xRz, i.e., $af = be$. (Know: $adcf \overset{(5)}{=} bcde$)

We discuss cases:

(i) if $dc \neq 0$, then we can divide both sides of (5) by dc to get $af = be$.

(ii) if $dc = 0$. then $c = 0$ since $d \neq 0$.

Since $ad = bc = b \cdot 0 = 0$ and $0 = 0 \cdot f = cf = de$ and $d \neq 0$.

It follows that $a = 0$ and $e = 0$. so we have $af = be = 0$.

By (1), (2), and (3), R is an equivalence relation. $\square$

Point: The equivalence classes of R in $A = \mathbb{Z}$ are exactly the rational numbers,

e.g. $\frac{3}{5} \Leftrightarrow \left[\frac{3}{5}\right] = \left\{\frac{6}{10}, \frac{-6}{-10}, \frac{30}{50}, \ldots \ldots\right\}$

so R gives us a way of "constructing the rational numbers from integers".

2. HW discussion

induction vs. strong induction :

difference → inductive step.

induction.

assume S_k and prove S_{k+1} using S_k alone

strong induction

assume all of $S_1, S_2, \ldots, S_k$ have all been proved and prove S_{k+1} using all of $S_1, \ldots, S_k$.