Math 2001. Lecture 34.

04. 18. 2022.

Last time: induction worksheet

Today: Ch. 11. Relations. $\to$ relations, equivalence relations.

1. Motivation, notation, and definition

Many mathematical statements assert some relationship between two objects:

$1 < 2$, $4 \leq 4$, $6 = \frac{30}{5}$, $3 \in \{1, 2, 3\}$

$a \equiv b \pmod{n}$, $X \subseteq Y$, $\mathbb{R} \not\subseteq \mathbb{Z}$, $3 \mid 18$, $3 \nmid 19$.

$3 R 18$ where $R =$ "divides" $\xrightarrow{\text{encode}}$ $(3, 18)$

Notation: All the above statements (indeed all statements) describing a relation between two objects x and y can be expressed as xRy. $\to$ "(x, y)"

Formally, we can encode relations via pairs as follows:

Def: (Formally) A relation on a set A is a subset $R \subseteq A \times A$.

We often abbreviate the statement $(x,y) \in R$ as $x R y$.

The statement $(x,y) \notin R$ is abbreviated as $x \not{R} y$.

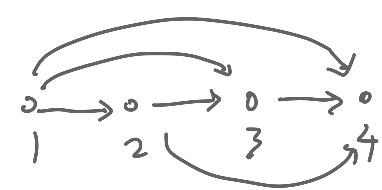
Sometimes, especially when A is finite, we can capture the relation R via a directed graph $G = (V, E)$ where the set of vertices is $V = A$ and where the edge set is $E = \{ (x,y) : x, y \in A, \ xRy \}$.

$x \to y$

E.g. $A = \{1, 2, 3, 4\}$

$R = \{ (1,2), (1,3), (1,4), (2,3), (2,4), (3,4) \}$

"<"



The following natural properties of relations are often interesting to study:

Def: Suppose R is a relation on a set A. We say that

(1) R is <u>reflexive</u> if xRx $\forall x \in A$.

(2) R is <u>symmetric</u> if $\forall x, y \in A$, $xRy \Rightarrow yRx$.

(3) R is <u>transitive</u> if $\forall x, y, z \in A$, xRy and $yRz \Rightarrow xRz$.

Example:

Relations on $A := \mathbb{Z}$.	$<$	$\leq$	$=$	$\mid$	$\nmid$	$\neq$
Reflexive	No ($1 \not< 1$)	Y	Y	Y	$n \nmid n$ $\forall n$? N	N
Symmetric	No ($1<2$, $2 \not< 1$)	N	Y	N	$a \nmid b \Rightarrow b \nmid a$? N	Y
Transitive	Yes	Y	Y	Y	$a \nmid b, b \nmid c \Rightarrow a \nmid c$? 2 3 4 N	N 1, 2, 1

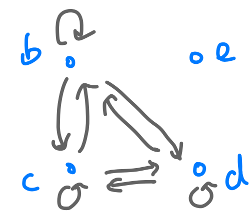
Example 2: Consider the following relation R on the set $A = \{b, c, d, e\}$.

$$R = \{(b,b), (b,c), (c,b), (c,c), (d,d), (b,d), (d,b), (c,d), (d,c)\}.$$

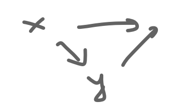
Is R

- reflexive? No, since $e \not R e$.
- symmetric? Yes, since $xRy \Rightarrow yRx$ $\forall x, y \in A$ by inspection.
- transitive? Yes, by careful, exhaustive inspection.

if we encode



In pictures: A relation R on a set A is

- reflexive if  $\forall a \in A$ in the graph for R.
- symmetric if anytime we have an edge $x \to y$ we have $x \rightleftarrows y$ ... .
- transitive if anytime we have $x \to y \to z$ we have 

2. Equivalence relations

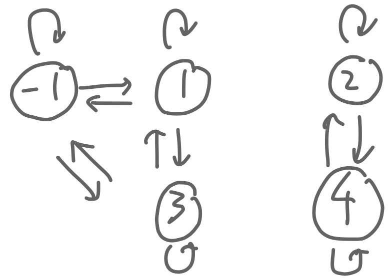
Def: A relation R on a set A is an equivalence relation if it is reflexive, symmetric, and transitive.

Def: Suppose R is an equivalence relation on a set A. For each elt $a \in A$, the equivalence class of / containing a is the set, denoted by $[a]$, given by

$$[a] := \{ x \in A : xRa \}.$$

Note: if R is an equiv. rel. on A, then by symmetry (since R is symmetric), we have $[a] = \{ x \in A : aRx \}$ $\forall a \in A$.

Examples: Consider the equivalence relation R on $A = \{-1, 1, 3, 2, 4\}$ given by the graph



(e.g. we would get this picture if R = "having the same parity".)

Note: $[a] = \{$ vertices b adjacent to a, i.e. vertices s.t. $a \to b$ is an edge $\}$
(equiv. $b \to a$ is an edge)

$[-1] = \{-1, 1, 3\}$, $[1] = \{-1, 1, 3\}$, $[3] = \{-1, 1, 3\}$

$[2] = \{2, 4\}$, $[4] = \{2, 4\}$.

classes: $\{-1, 1, 3\}$, $\{2, 4\}$

Observation: (*) while there are 5 elts in A, there are only 2 distinct equivalence classes for R; the classes partition A, and they correspond to the connected components of the graph for R.

Note: If R is a rel on A that's not an equiv rel.

the sets $[a] := \{ x \in A : x R a \}$ don't behave as nicely as the

equiv. classes do when R is an equiv class.

Next time: · examples

· generalizing (*)